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ANALYSIS OF ECOLOGICAL MODEL WITH DIFFUSION REACTION AND HOLLING TYPE FUNCTIONAL RESPONSE

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ABSTRACT

The paper presents a comprehensive study of the complex dynamics of plant-Herbivores animal-carnivore model. This study established the behavior; dynamics of the 3-participating species along with the Holling Type-II & Type-IV functional responses. The local, global stability and Hopf-bifurction analysis of the system are discussed. Further, we incorporated diffusion in to the P-H-C model and evaluated it's stability and then derived the effect of diffusion and instability conditions and also discussed the global stability of diffusion version of the model system. The study has presented numerical illustrations to scientifically support the findings of the analysis.

Keywords: *Plants, Herbivores, carnivores, Local stability, Global stability, Hopf bifurcation, Reaction-Diffusion, Turing instability*.

1. INTRODUCTION

There is a dynamic and non-linear existence of ecological processes such as Prey- Predator, Food-Chain structure, etc. One of the significant topics of ecology is the dynamics of the food-chain system. Most of the researchers extended their studies to better understand the complex hierarchical behavior of ecological systems in real world,many researchers turned their interests to ecological models of two or more organisms. Several of the ecological models taken into account in ecological literature [11-19] are built on functional responses. In food chain models, functional responses play a vital role. Chaotic dynamics have been found in much literature in 3 species of models with Type-I and Type-II functional reactions. Authors like Ranjith Kumar Upadhyay [11,13,16,17,26,29] were proposed the three species models with Holling Type-II and Type-IV functional responses and analyzed the behavioral dynamics around the co-existing state and also discussed the bifurcation analysis around the co-existing state.

Individual species are separated from a biological point of view in environment and generally interact with physical world and other organisms in their structural neighborhoods. It is very feasible to study diffusion mechanism in the models of biological, physical and chemical systems. FengRao,[28,36] stated that Pure Diffusion Mechanism generally results in stabilizing effect in these systems. This makes the system seems in a stable state and diffusion can also contribute to predator extinction in some cases.Self-diffusion plays a predominant role in ZhifuXiemodel [25], with its three species model and instability driven by the intra-species diffusion and intra-species interactions. In his theoretical analysis, researchers such as David [19, 26] suggest that the feeding strategies of predators may deter the development of Turing spatial patterns. We were motivated to do this paper by all these studies.

The paper presents complex behavior model of food chain (P-H-C) with functional responses of Holling type-II & type-IV without diffusion and with diffusion. In our proposed model, plants are primary food producers for herbivores and herbivores are food producers for carnivores. So, herbivores act as prey for carnivores and predators for plants. For the study of the interaction species, Diffusion models provide a reasonable basis. By using reaction diffusion equations, we formulated a diffusion

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version of system and discussed the stability of system in a coexisting steady state, as well as the conditions of Turing instability. The proposed model is as follows

2. THE FOOD CHAIN MODEL

In this paper, the plants act as resource biomass. The herbivores grow by eating plants and a carnivore grows by eating herbivores. In this model of Plants-Herbivores-Carnivores (P-H-C), plants are purely prey. Herbivores are food source for carnivores and simultaneously food source for herbivores is plants population. Here herbivore acts as both prey and predator simultaneously. The 3 species food chain model can be mathematically interpreted by the following differential equation system:

$$\begin{aligned} \frac{dP}{dt} &= rP\left(1-\frac{P}{k}\right)-\frac{\beta_1 PH}{\alpha_1+P} \\ \frac{dH}{dt} &= sH\left(1-\frac{H}{l}\right)+\frac{\beta_2 PH}{\alpha_1+P}-\frac{\gamma_1 HC}{\alpha_2+H^2} \\ \frac{dC}{dt} &= \frac{\gamma_2 HC}{\alpha_2+H^2}-\beta_0 C. \end{aligned} \tag{2.1}$$

Where P(t), H(t) and C(t) denote the density of plants, herbivores and carnivores respectively at any instant of time t. $r,s,k,l,\alpha_1,\alpha_2,\beta_0,\beta_1,\beta_2,\gamma_1,\gamma_2$ are positive constants. The parameters r and s are intrinsic growth rates of plants and herbivores respectively, and it is also assumed that the growth of the plants and herbivores is logistic. The parameters β_1 and γ_1 denote the attack rate at which the single herbivores searches for plant and the single carnivores searches for herbivores, whenever predator is not currently consuming prey item. The parameters α_1 and α_2 are half saturation levels of Herbivores and carnivores respectively. The parameters k and l are carrying capacities of plant populations and herbivores population respectively. The parameter β_0 is the mortality rate of the carnivores.

The system (2.1) has eleven parameters. It is evident that dealing with a system having more number of parameters is challenging and requires more complicated analysis. Reformulating a model in dimensionless type is helpful in many aspects. This procedure will facilitate to see the consistency of the model equations and ensure that each one terms have an equivalent set of units in equation.In addition, the non-dimensionalization of the model reduces the amount of parameters and divulges a smaller number of parameters to control the complexities.

After non-dimensionalization, theabove form of system (2.1) is

$$\begin{aligned} \frac{dP}{dt} &= \alpha P - P^2 - \frac{\beta PH}{\gamma+P} \\ \frac{dH}{dt} &= \delta H - H^2 + \frac{\lambda PH}{\gamma+P}-\frac{HC}{\mu+H^2} \\ \frac{dC}{dt} &= \frac{\rho HC}{\mu+H^2}-C \end{aligned} \tag{2.2}$$

Where $\alpha=\frac{r}{\beta_0},\beta=\frac{\beta_1 l}{s},\gamma=\frac{\alpha_1 r}{\beta_0 k},\delta=\frac{s}{\beta_0},\lambda=\frac{\beta_2 k}{r},\mu=\frac{\alpha_2 s}{\beta_0 l},\rho=\frac{\gamma_2 l}{s}.$

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Through the governing framework of non-linear diffusion equations, continuous population distributions that undergo self-diffusion and interaction in a three-dimensional space are dynamically defined. The reaction diffusion equations are widely used in ecology as models of spatial effects. In the study, we included diffusion for both the predator and prey. The diffusion version of the above system (2.2) is

$$\begin{aligned}
&\frac{\partial P(u,t)}{\partial t} - D_1.\Lambda P = \alpha P - P^2 - \frac{\beta PH}{\gamma + P};\\
&\frac{\partial H(u,t)}{\partial t} - D_2.\Lambda H = \delta H - H^2 + \frac{\lambda PH}{\gamma + P} - \frac{HC}{\mu + H^2};\\
&\frac{\partial C(u,t)}{\partial t} - D_3.\Lambda C = \frac{\rho HC}{\mu + H^2} - C;\\
&\frac{\partial P(u,t)}{\partial u} = \frac{\partial H(u,t)}{\partial u} = \frac{\partial C(u,t)}{\partial u} = 0; \quad u \in \partial\Omega, t > 0.
\end{aligned} \tag{2.3}$$

Where $P = P(u,t),\ H = H(u,t),\ C = C(u,t)$ denote the Plant, Herbivore, Carnivore population densities in the habitat Ω. The diffusion symbol $\Lambda = \frac{\partial^2}{\partial^2 u}$ denotes the Laplacian operator, $\partial\Omega$ is the smooth boundary, $\frac{\partial}{\partial u}$ is outward directional derivative normal to $\partial\Omega$ and positive constants D_1, D_2, D_3 are diffusion co-efficient of P, H, C respectively.

3. STABILITY ANALYSIS

This section presents the behavior of proposed system with a diffusion model and without a diffusion model.

3.1. WITHOUT DIFFUSION

It is observed that the R.H.S of the system (2.2) is continuous and contains continuous partial derivatives on non-negative state space $R_+^3 = \left\{(P,H,C) \in R^3 : P \ge 0, H \ge 0, C \ge 0\right\}$.Therefore, the system (2.2) admits a unique solution in R_+^3 and it is uniformly bounded. The following theorem proves that all the species are bounded evenly for any of the initial value in R_+^3.

Theorem (3.1.1) the system (2.2) is bounded uniformly.

Proof: From the first equation of system (2.2), we have $\frac{dP}{dt} \le P(\alpha - P)$,from that we obtain $P(t) \le \alpha.\left(1 + \frac{(\alpha - P_0)}{P_0}.e^{-\alpha t}\right)^{-1}$ $(\mathit{for\ all}\, t \ge 0)$. For large value of t, this inequality this inequality becomes $P(t) \le \alpha,\ \mathit{for\ all}\ t \ge 0$. Now, we define $\Upsilon(t) = \frac{\lambda}{\beta}P(t) + H(t) + \frac{1}{\rho}C(t)$, then $\frac{d\,\Upsilon(t)}{dt} \le \frac{(\alpha^2 + \alpha)}{\beta} + \eta - \Upsilon(t)$ (Where $\eta = \min\left\{1, \lambda, \delta\right\}$).From that, we obtain

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$\Upsilon(t)\le\left(\frac{(\alpha^2+\alpha)}{\beta}+\eta\right)+e^{-t}.\left(\frac{(\alpha^2+\alpha)}{\beta}+\eta-\Upsilon(0)\right)$ *for* $t\ge 0$. (By comparison theorem). Thus,

$\Upsilon(t)=\left(\frac{\lambda}{\beta}P(t)+H(t)+\frac{C(t)}{\rho}\right)\le\left(\frac{(\alpha^2+\alpha)}{\beta}+\eta\right)$ *for* $t\ge 0$. Hence, the system (2.2) is bounded Uniformly for anyinitialvalue in R_+^3.

The system has the following six steady states

1). $E_0=(0,0,0)$. 2). $E_1=(0,\delta,0)$ 3). $E_2=(\alpha,0,0)$ 4). $E_3=(0,\bar{H},\bar{C})$ where

$\bar{H}=\frac{\rho\pm\sqrt{\rho^2-4\mu}}{2},\left(\rho^2>4\mu\right)$ and $\bar{C}=\frac{\rho}{2}\left((\delta-\rho)\left(\rho\pm\sqrt{\rho^2-4\mu}\right)-2\mu\right)$.

5). $E_4=(\hat{P},\hat{H},0)$ where $\hat{H}=\delta+\frac{\lambda\hat{P}}{\gamma+\hat{P}}$ and $\hat{P}=\frac{1}{2}\left[\left(\alpha-\gamma\right)\pm\sqrt{\left(\alpha-\gamma\right)^2-4\left(\beta\hat{H}-\alpha\gamma\right)}\right]$.

6). The positive steady state $E_5=(P^*,H^*,C^*)$, where $H^*=\frac{\rho\pm\sqrt{\rho^2-4\mu}}{2}$,

$P^*=\frac{1}{2}\left[\left(\alpha-\gamma\right)\pm\sqrt{\left(\alpha+\gamma\right)^2-4\beta H^*}\right]$ And $C^*=H^*\rho\left[(\delta-H^*)+\frac{\lambda P^*}{\gamma+P^*}\right]$.

Theorem (3.1.2) The interior equilibrium point $E_5=(P^*,H^*,C^*)$ exists, if $\rho^2>4\mu,\left(\alpha+\gamma\right)^2>2\beta H^*$ and $\lambda>\left(H^*-\delta\right)\left(\frac{\gamma+P^*}{P^*}\right)$.

Proof:-Let P^*,H^*,C^* are positive solutions of the following equations

$\alpha-P^*-\frac{H^*\beta}{\gamma+P^*}=0,\delta-P^*+\frac{P^*\lambda}{\gamma+P^*}-\frac{C^*}{\mu+H^{*2}}=0,\frac{H^*\rho}{\mu+H^{*2}}-1=0.$

By solving these equations we obtain

$H^*=\frac{\rho\pm\sqrt{\rho^2-4\mu}}{2},P^*=\frac{1}{2}\left[\left(\alpha-\gamma\right)\pm\sqrt{\left(\alpha-\gamma\right)^2-4\left(\beta H^*-\alpha\gamma\right)}\right]$ *and* $C^*=H^*\rho\left[(\delta-H^*)+\frac{\lambda P^*}{\gamma+P^*}\right]$.

Hence, the interior steady state $E_5=(P^*,H^*,C^*)$ exists, if $\rho^2>4\mu$, $\left(\alpha+\gamma\right)^2>2\beta H^*$ and $\lambda>\left(H^*-\delta\right)\left(\frac{\gamma+P^*}{P^*}\right)$.

Now, we can discuss local stability of the coexistent state point $E_5=(P^*,H^*,C^*)$ as follows

Theorem (3.1.3) The interior steady state $E_5=(P^*,H^*,C^*)$ is stable locally, if $B_1>0,B_3>0$ and $\left(B_1B_2-B_3\right)>0$, otherwise, it is unstable.

Proof:- For the point $E_5=(P^*,H^*,C^*)$ the corresponding variational matrix is

$$J\left(E_5\right)=\begin{pmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & b_{23} \\ 0 & b_{32} & 0 \end{pmatrix},$$ Where

$$b_{11}=\frac{H^*P^*\beta}{\left(\gamma+\overset{*}{P}\right)^2}-\overset{*}{P}, b_{12}=\frac{-P^*\beta}{\gamma+-P^*}, b_{21}=\frac{H^*\lambda\gamma}{\left(\gamma+P^*\right)^2}, b_{22}=-H^*+\left[2.C^*.H^{*2}\right]\left[\left(\mu+H^{*2}\right)^2\right]^{-1},$$

$$b_{23}=-H^*.\left[\mu+H^{*2}\right] \quad , b_{32}=C^*.\rho.\left[\mu-H^{*2}\right].\left[\left(\mu+H^{*2}\right)^2\right]^{-1}$$

The Eigen equation of $J\left(E_5\right)$ is $\lambda^3+B_1\lambda^2+B_2\lambda+B_3=0$.

Here $B_1=-\left(b_{11}+b_{22}\right), B_2=\left(b_{11}b_{22}-b_{32}b_{23}-b_{12}b_{21}\right), B_3=b_{11}b_{32}b_{23}$.

Now

$B_1=\frac{H^*}{N_2^2N_3^2}\left[N_1+\frac{P^*}{H^*}N_2^2N_3^2\right]; B_3=\frac{P^*H^*C^*\mu^2\rho\left(\mu-H^{*2}\right)}{N_2^3}\left(\frac{N_3^2-H^*\beta}{N_3^3}\right)$; and N_1, N_2, N_3 are defined by

$N_1=N_2^2N_3^2-\left(2H^*C^*N_2^2+P^*\beta N_3^2\right), N_3=\left(\mu+H^{*2}\right)>0$, $N_2=\left(\gamma+P^*\right)>0$.

If $\beta<\frac{N_2^2\left(N_3^2-2C^*H^*\right)}{P^*N_3^2}, \beta<\frac{N_2^2}{H^*}$, the coefficients of characteristic equation $B_1>0$ and $B_3>0$. Again, consider $\Delta=B_1B_2-B_3=\left(b_{11}+b_{22}\right)\left(b_{11}b_{22}-b_{32}b_{23}-b_{12}b_{21}\right)-\left(b_{22}b_{32}b_{23}\right)$.

$$\Delta=\frac{P^*H^*}{N_2^4N_3^2}\left\{N_2^2-\left(H^*\beta\right).\left[\left[N_1H^*+P^*N_2^2N_3^2\right]\left(N_3^2-2C^*H^*\right)-C^*\rho N_2^2N_3\left(\mu-H^{*2}\right)\right]+\left[N_1H^*+P^*N_2^2N_3^2\right]\left(\frac{\beta\gamma\lambda N_3^2}{N_2}+\frac{C^*\rho N_2^2\left(\mu-H^{*2}\right)}{N_3P^*}\right)\right\}.$$

That implies, $\Delta=B_1B_2-B_3>0$ if

$$\lambda>\frac{N_2}{N_3^2\gamma\beta}\left\{\left(C^*\rho N_2^2\left(\mu-H^{*2}\right)\right).\left[\frac{\left(N_2^2-H^*\beta\right)N_3}{\left[N_1H^*+P^*N_2^2N_3^2\right]}-\frac{1}{N_3P^*}\right]-\left(N_2^2-H^*\beta\right)\left(N_3^2-2C^*H^*\right)\right\}$$

Therefore, By Routh-Hurwitze criteria, the interior steady state $E_5=(P^*,H^*,C^*)$ is locally asymptotically stable, if. $B_1>0, B_3>0$ And $\left(B_1B_2-B_3\right)>0$.

Theorem (3.1.4) Along with the conditions stated in the theorems (**3.1.2**),(**3.1.3**) and **if** $\beta H<\left(\gamma+P\right)\left(\gamma+P^*\right) \& \rho^2>\frac{\left(H+H^*\right)}{H.H^*}$ Then, the steady state $E_5=(P^*,H^*,C^*)$ is globally Asymptotically stable.

Proof: Let's consider positive definite function

$$V_1\left(P,H,C\right)=n_1\left[P-P^*-P^*\ln\left(\frac{P}{P^*}\right)\right]+n_2\left[H-H^*-H^*\ln\left(\frac{H}{H^*}\right)\right]+n_3\left[C-C^*-C^*\ln\left(\frac{C}{C^*}\right)\right],$$

where n_1, n_2 and n_3 are positive constants to be determined.

$$\frac{dV_1}{dt}=n_1\left[\frac{\beta H^*}{\left(\gamma+P\right)\left(\gamma+P^*\right)}-1\right]\left[P-P^*\right]^2+n_2\left[\frac{C^*\left(H+H^*\right)}{\left(\mu+H^{*2}\right)\left(\mu+H^2\right)}-1\right]\left[H-H^*\right]^2+$$

$$\frac{1}{\left(\gamma+P\right)}\left[n_2\lambda-n_1\beta-\frac{n_2\lambda P^*}{\left(\gamma+\overset{*}{P}\right)}\right]\left(P-P^*\right)\left(H-H^*\right)+\frac{1}{\left(\mu+H^2\right)}\left[n_3\rho-n_2-\frac{n_3\rho H^*}{\left(\mu+H^{*2}\right)}\right]\left(C-C^*\right)\left(H-H^*\right)$$

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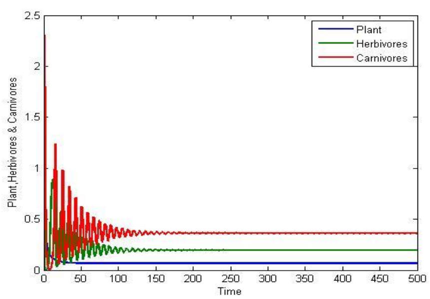
Chose non-negative constants

$n_1 = n_2 = 1, n_3 = \frac{H^*}{\mu - HH^*}, \lambda = \left(\frac{\beta(\gamma + P^*)}{\gamma}\right)$ and if $\beta H < (\gamma + P)(\gamma + P^*) \,\&\, \rho^2 > \frac{C^*(H + H^*)}{H.H^*}$,then $\frac{dV_1}{dt} < 0$. Therefore by Lyapunov theorem the steady state $E_5 = (P^*, H^*, C^*)$ is globally asymptotically stable.

NUMERICAL SIMULATIONS

If choose the parameter values $\alpha = 0.069$; $\beta = 0.00025$; $\gamma = 0.0985$; $\delta = 1$; $\lambda = 0.002$; $\rho = 2.3$; $\mu = 0.412$;Then we obtain The positive Steady state point (P^*, H^*, C^*)=(0.0687, 0.195, 0.3125) and $B_1 = 0.3125 > 0$;$B_3 = 0.0458 > 0$; $B_1B_2 - B_3 = 0.0397 > 0$.Hence, the system (2.2) is stable at the point (0.0687, 0.195, 0.3125). The corresponding stable graphs as follows



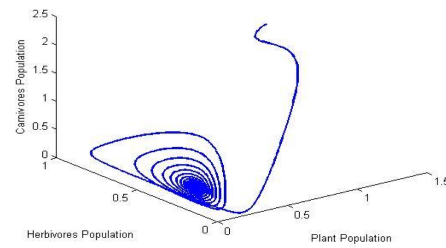
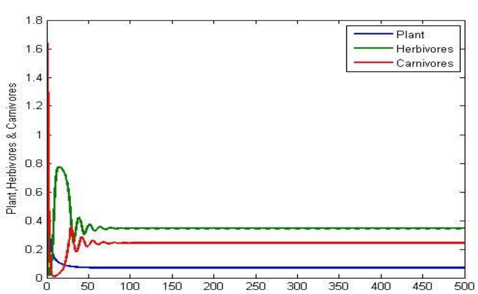


Fig.1

Fig.2

If choose the parameter values $\alpha = 0.069$; $\beta = 0.00025$; $\gamma = 0.985$; $\delta = 0.8$; $\lambda = 0.0312$; $\rho = 1.538$; $\mu = 0.412$;Then we obtain The positive Steady state point (P^*, H^*, C^*)=(0.0689, 0.3448, 0.2428) and $B_1 = 0.2089 > 0$;$B_3 = 0.0174 > 0$; $B_1B_2 - B_3 = 0.0374 > 0$.Hence, the system (2.2) is stable at the point (0.0689, 0.3448, 0.2428) .The corresponding stable graphs as follows



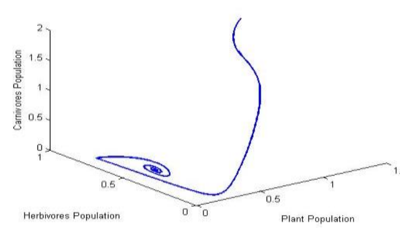


Fig.3

Fig.4

If choose the parameter values $\alpha = 0.0095$; $\beta = 0.0009$; $\gamma =$ 0.0986; $\delta = 1.01$; $\lambda = 0.78$; $\rho = 2.1$; $\mu =$ 0.48 ; The corresponding stable graphs as follows

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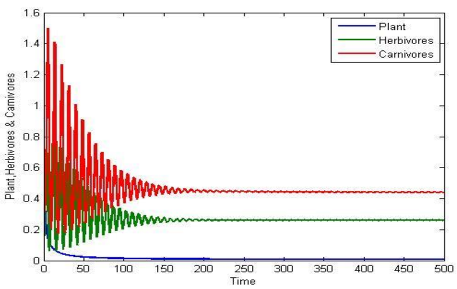

Fig.5

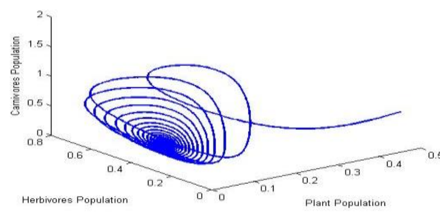

Fig.6

If choose the parameter values $\alpha = 0.0095$; $\beta = 0.0009$; $\gamma = 0.0986$; $\delta = 1.01$; $\lambda = 0.0000000001$; $\rho = 2.1$; $\mu = 0.48$; The corresponding stable graphs as follows

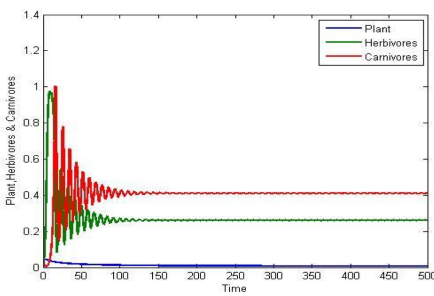

Fig.7

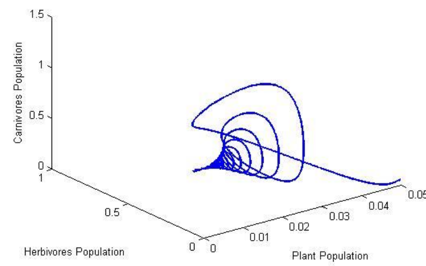

Fig.8

If choose the parameter values $\alpha = 0.1$; $\beta = 0.0009$; $\gamma = 0.0986$; $\delta = 1.01$; $\lambda = 0.0001$; $\rho = 2.1$; $\mu = 0.48$; The corresponding stable graphs as follows

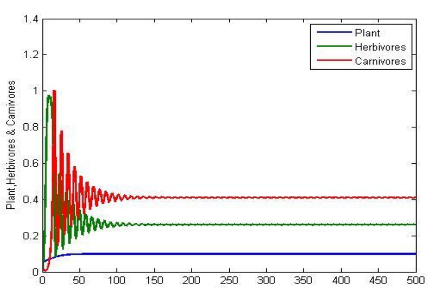

Fig.9

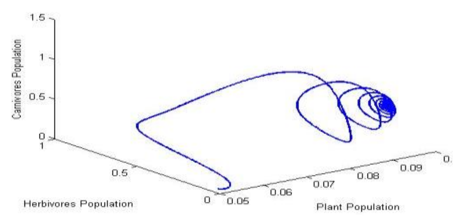

Fig.10

HOPF BIFURCATION

In this model, several parameters are used to describe the dynamical system's behavior. If any parameters values used in the model are altered, the behavior of the system will also change. The sensitive values with such types of transitions are commonly referred to as bifurcation points. Hopf bifurcation emerges at those points where the system has non-trivial periodic solutions. In this section, we established that Hopf occurs for the system (2.2) at a critical value $\lambda = \lambda^*$.

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Theorem (3.1.5) Assume that $\beta < \frac{\left(N_2{}^2 - C^*\right)}{P^* N_2{}^2}$ and $\beta < \frac{N_3{}^2}{H^*}$, holds then the simple Hopf bifurcation of the system (2.2) emerges at $\lambda = \lambda^*$.

Proof: Assume that the theorem (3.1.3) conditions holds, and

Let $\lambda^* = \frac{N_2}{N_3^2 \gamma\beta}\left\{\left(C^* \rho N_2^2\left(\mu - H^{*2}\right)\right).\left[\frac{\left(N_2^2 - H^*\beta\right)N_3}{\left[N_1 H^* + P^* N_2^2 N_3^2\right]} - \frac{1}{N_3 P^*}\right] - \left(N_2^2 - H^*\beta\right)\left(N_3^2 - 2C^* H^*\right)\right\}$.

Then

$$B_1\Big|_{\lambda=\lambda^*} = \frac{H^*}{N_2^2 N_3^2}\left[N_1 + \frac{P^*}{H^*} N_2^2 N_3^2\right] > 0,\; B_3\Big|_{\lambda=\lambda^*} = \frac{P^* H^* H^* \mu^2 \rho\left(\mu - H^{*2}\right)}{N_2^3}\left(\frac{N_3^2 - H^*\beta}{N_3^3}\right) > 0$$

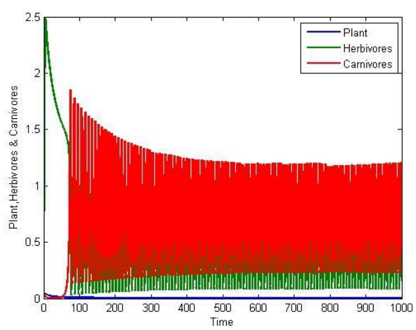
And

$\frac{d\Delta}{d\lambda}\Big|_{\lambda=\lambda^*} = \frac{P^* H^* \beta\gamma}{N_2^5 N_3^2}\left[N_1 H^* + P^* N_2^2 N_3^2\right] \neq 0$. Therefore, $\frac{d\Delta}{d\lambda}\Big|_{\lambda=\lambda^*} \neq 0$.

Hence, a simple Hopf bifurcation occurs at $\lambda = \lambda^*$.

NUMERICAL SIMULATIONS

Using Theorem (3.1.5) we have determined the critical value of λ as $\overset{*}{\lambda}$ =2.95. The system is found to be unstable for $\lambda > \overset{*}{\lambda}$ around the positive steady state. The instability graphs as follows



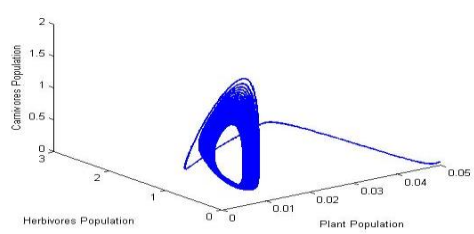


Fig. 10 Fig.11 Bifurcation diagram for $\lambda = 0.39$
$\alpha = 0.0089$; $\beta = 0.0009$; $\gamma = 0.985$; $\delta = 1.01$; $\rho = 2.1$; $\mu = 0.48$

3.2. WITH DIFFUSION

This section presents a detailed study on dynamics of diffusion version of the system (2.3).

Let $\Phi = \left(P, H, C\right)^T$, $D = diag(D_1, D_2, D_3)$,

$F(\Phi) = \left(\alpha P - P^2 - \frac{\beta PH}{\gamma + P}, \delta H - H^2 + \frac{\lambda PH}{\gamma + P} - \frac{HC}{\mu + H}, \frac{\rho HC}{\mu + H} - C\right)^T$, Then the above system (2.3) can be presented as

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$$\begin{cases} \dfrac{\partial \Phi}{\partial t} = D.\Lambda\Phi + F(\Phi), \\ \dfrac{\partial \Phi}{\partial u} = 0, \quad \Phi(u,0) = \left(P_0(u), H_0(u), C_0(u)\right)^T \end{cases}$$

Let $O = \mu_0 < \mu_1 < \mu_2 < \mu_3 <$.........be the Eigen values of the operator $-\Lambda$ in Ω with a boundary condition at zero flux, $\mathrm{E}(\mu_k)$ be the Eigen space corresponding to μ_k in $C^1(\Omega)$.

Let $\{\varphi_{kj} \mid j = 1,2,3,\ldots \dim \mathrm{E}(\mu_k)\}$ be a group of orthonormal Basis of $\mathrm{E}(\mu_k)$.

$$Y = \left\{\Phi = (P,H,C)^T \in \left(C^1(\Omega)\right)^3; \frac{dP}{du} = \frac{dH}{du} = \frac{dC}{du} = 0 \quad on \quad \partial\Omega\right\},$$

$Y_{kj} = \{c\varphi_{kj} : c \in R^3\}$, then $Y = \bigoplus_{k=0}^{\infty} Y_k$, (where $Y_k = \bigoplus_{j=1}^{\dim \mathrm{E}(\mu_k)} Y_{kj}$)

In the positive steady state $E_5 = (P^*, H^*, C^*)$, the linearization of the system (2.3) can be represented by $\Phi_t = L(\Phi) \equiv D.\Lambda\Phi + J_0(\Phi)$, Where

$$J_0 = J\left(E_5\right) = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & 0 \end{pmatrix}, \text{Here}$$

$$a_{11} = -P^* + \frac{P^*H^*\beta}{\left(\gamma + P^*\right)^2}, a_{12} = \frac{-\beta P^*}{\gamma + P^*}, a_{21} = \frac{\lambda\gamma P^*}{\left(\gamma + P^*\right)^2}, a_{22} = 2C^*P^{*2}.\left[\left(\mu + H^{*2}\right)^2\right]^{-1} - H^*,$$

$$a_{23} = \frac{-H^*}{\left[\mu + H^{*2}\right]} \quad , a_{32} = C^*\rho\left(\mu - H^{*2}\right).\left[\left(\mu + H^{*2}\right)^2\right]^{-1}$$

Under the operator L, each $k \geq 0, Y_k$ is invariant and λ is the Eigen value of L if and only if the Eigen value of the matrix $J_* = -\mu_k^i D + J_0, \left(\mu_k^i D = D_i k^2\right) i = 1,2,3$ for some $k \geq 0$,then

$$J_*(P^*, H^*, C^*) = \begin{pmatrix} a_{11} - D_1k^2 & a_{12} & 0 \\ a_{21} & a_{22} - D_2k^2 & a_{23} \\ 0 & a_{32} & -D_3k^2 \end{pmatrix}, \text{(Here k is the Wave number.)}$$

The stability characteristics of the attracting coexisting equilibrium point are our concern that will leads to the conditions of Turing instability. From above matrix, we get the Eigen equation of the form

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$$\det\left(J_* - \lambda I\right) = \lambda^3 - Tr\left(J_*(P^*, H^*, C^*)\right)\lambda^2 + \begin{bmatrix} (D_1D_2 + D_2D_3 + D_3D_1)k^4 \\ -\left\{a_{11}(D_2 + D_3) + a_{22}(D_1 + D_3)\right\}k^2 \\ + a_{11}a_{22} - a_{12}a_{21} - a_{23}a_{32} \end{bmatrix}\lambda$$
$$- \det\left(J_*(P^*, H^*, C^*)\right) = 0,$$

Where $Tr\left(J_*(P^*, H^*, C^*)\right) = \left\{a_{11} + a_{22} - (D_1 + D_2 + D_3)k^2\right\},$

$$-\det\left(J_*(P^*, H^*, C^*)\right) = a_{11}a_{32}a_{23} + k^2\left\{(a_{11}a_{22} - a_{12}a_{21})D_3 - a_{23}a_{32}D_1\right\}$$
$$-k^4(a_{11}D_2 + a_{22}D_1)D_3 + k^6D_1D_2D_3.$$

The Eigen equation J_* can be written as $\lambda^3 + \delta_1\lambda^2 + \delta_2\lambda + \delta_3 = 0$, where

$$\delta_1 = -Tr\left(J_*(P^*, H^*, C^*)\right),$$
$$\delta_2 = (D_1D_2 + D_2D_3 + D_3D_1)k^4 - \left\{a_{11}(D_2 + D_3) + a_{22}(D_1 + D_3)\right\}k^2 + (a_{11}a_{22} - a_{12}a_{21} - a_{23}a_{32}),$$
$$\delta_3 = -\det\left(J_*(P^*, H^*, C^*)\right).$$

The following theorem has been discussed based on the Routh-Hurwitze criteria

Theorem (3.2.1) The presence of diffusion in the system at the coexisting steady state point is stable locally only if the following conditions are met:

$$a_{11} + a_{22} < (D_1 + D_2 + D_3)k^2 \qquad (i)$$
$$k^6D_1D_2D_3 - k^4(a_{11}D_2 + a_{22}D_1)D_3 + k^2\left\{(a_{11}a_{22} - a_{12}a_{21})D_3 - a_{23}a_{32}D_1\right\}$$
$$+a_{11}a_{32}a_{23} > 0. \qquad (ii)$$
$$(D_1D_2 + D_2D_3 + D_3D_1)k^4 - \left\{a_{11}(D_2 + D_3) + a_{22}(D_1 + D_3)\right\}k^2$$
$$+ a_{11}a_{22} - a_{12}a_{21} - a_{23}a_{32} < \frac{\det\left(J_*(P^*, H^*, C^*)\right)}{Tr\left(J_*(P^*, H^*, C^*)\right)}. \qquad (iii)$$

DIFFUSION INSTABILITY

The conditions for the Turing instability have been calculated in the present section. Provided at least one Eigen value of the Eigen equation is positive, the system would be unstable. This will happen when the above theorem (3.2.1) does not contain at least one of the three inequalities.

For above equation (i), since D_1, D_2, D_3 *and* k^2 are positive, the equation (i) always holds as $a_{11} + a_{22} < 0$ from the stability condition of coexistent steady state. Therefore, the inequality (ii) can be reversed, that is

$$h(k) = k^6D_1D_2D_3 - k^4(a_{11}D_2 + a_{22}D_1)D_3 + k^2\left\{(a_{11}a_{22} - a_{12}a_{21})D_3 - a_{23}a_{32}D_1\right\} + a_{11}a_{32}a_{23} < 0.$$

Where $h(k)$ can be written as

$$H(k^2) = (k^2)^3D_1D_2D_3 - (k^2)^2(a_{11}D_2 + a_{22}D_1)D_3 + k^2\left\{(a_{11}a_{22} - a_{12}a_{21})D_3 - a_{23}a_{32}D_1\right\} + a_{11}a_{32}a_{23} < 0.$$

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Minimum value of $H(k^2)$ falls at $k^2 = k_c^2$ and which is given by

$$k_c^2 = \frac{1}{3D_1D_2D_3} \times \left[(a_{11}D_2 + a_{22}D_1)D_3 + \left\{ (a_{11}D_2 + a_{22}D_1)^2 D_3^{\,2} - 3D_1D_2D_3 \times \left\{ (a_{11}a_{22} - a_{12}a_{21})D_3 - a_{23}a_{32}D_1 \right\} \right\}^{1/2} \right]$$

Therefore, $H(k_c^{\,2}) < 0$ is the required diffusive instability condition.

Theorem (3.2.2) For the model system (2.3), the criterion for diffusive instability is taken from critical wave number k_c of the first perturbations induced by the solution of the above equation.

Theorem (3.2.3) Along with the conditions stated in the theorem (3.2.1) andif (i) $\rho^2 > \dfrac{C^*(H - H^*)}{HH^*}$ (ii) $\gamma = \dfrac{\beta P^*}{\lambda - \beta}$ (iii) $(\alpha - \gamma) < \left(P + P^*\right)$ then , the steady state $E_5 = (P^*, H^*, C^*)$ is globally asymptotically stable.

Proof: Let $W(P,H,C) = L_1 \int \dfrac{\left(P - P^*\right)}{P} dP + L_2 \int \dfrac{\left(H - H^*\right)}{H} dH + L_3 \int \dfrac{\left(C - C^*\right)}{C} dC$ and

$\overset{*}{E}(t) = W\left[P(x,t), H(x,t), C(x,t)\right]$, Then $\dfrac{d\overset{*}{E}(t)}{dt} = \displaystyle\int_\Omega (W_P P_t + W_H H_t + W_C C_t)dx$

$$\frac{d\overset{*}{E}(t)}{dt} = -\int_\Omega \left[L_1 D_1 \frac{P}{P^*} |\nabla P|^2 + L_2 D_2 \frac{H}{H^*} |\nabla H|^2 + L_3 D_3 \frac{C}{C^*} |\nabla C|^2 \right] dx + \int_\Omega \left[-L_1 + \frac{L_1 \beta H^*}{(\gamma + P)(\gamma + P^*)} \right] \left(P - P^*\right)^2 dx +$$

$$\int_\Omega \left[\frac{L_2 C^*(H + \overset{*}{H})}{\left(\mu + H^{*2}\right)\left(\mu + H^2\right)} - L_2 \right] \left(H - H^*\right)^2 dx + \int_\Omega \frac{1}{(\gamma + P)} \left[\left(\frac{-L_2 \lambda P^*}{\left(\gamma + P^*\right)} \right) + \lambda L_2 - \beta L_1 \right] \left(P - P^*\right)\left(H - H^*\right) dx +$$

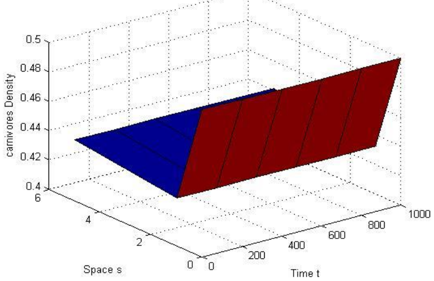
$$\int_\Omega \frac{1}{\left(\mu + H^2\right)} \left[\left(-\frac{L_3 H^* \rho (H + H^*)}{\left(\mu + H^{*2}\right)} \right) - L_2 + L_3 \rho \right] \left(C - C^*\right)\left(H - H^*\right) dx.$$

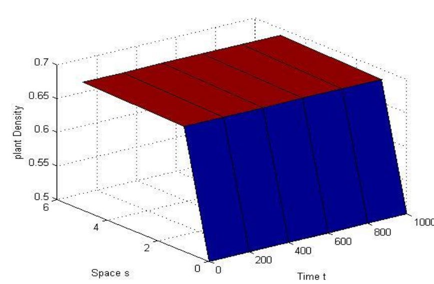
If we choose $L_1 = L_2 = 1, L_3 = \dfrac{H^*}{\left(\mu - HH^*\right)}, \gamma = \dfrac{\beta P^*}{\lambda - \beta}, (\alpha - \gamma) < \left(P + P^*\right)$ and $\rho^2 > \dfrac{C^*(H - H^*)}{HH^*}$,then

$$\frac{d\overset{*}{E}(t)}{dt} \le 0 .$$

Thus, if the above conditions satisfied, then the steady state $E_5 = (P^*, H^*, C^*)$ is globally asymptotically stable.

NUMERICAL SIMULATIONS





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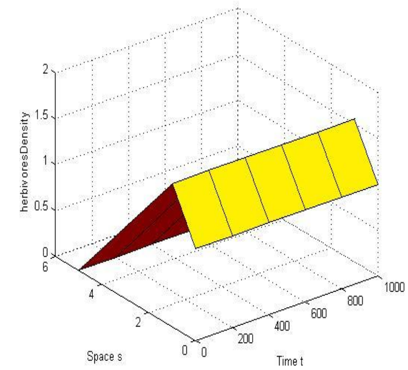


Fig.12 $\alpha = 0.69; \beta = 0.0025; \gamma = 0.0985; \delta = 1; \lambda = 0.8; \rho = 1.1; \mu = 0.312; D_1 = 0.051; D_2 = 0.05; D_3 = 1.1;$

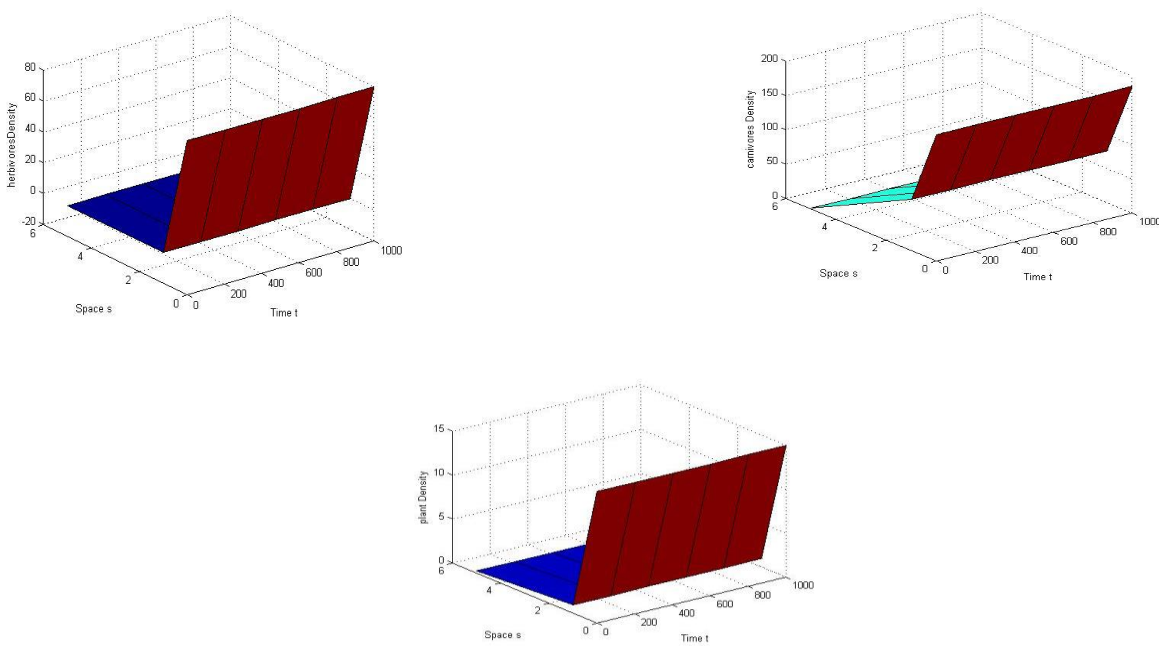
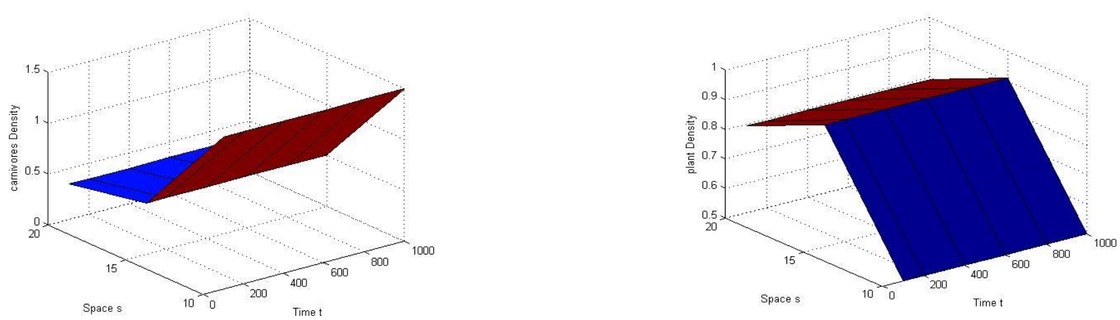


Fig: 13 $\alpha = 0.041; \beta = 0.003; \gamma = 2.985; \delta = 1; \lambda = 0.9; \rho = 2.3; \mu = 0.41; D_1 = 0.1; D_2 = 0.03; D_3 = 0.00000009;$



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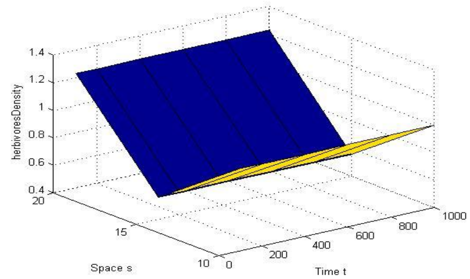

Fig: 14 $\alpha = 1; \beta = 1.95; \gamma = 10; \delta = 1.38; \lambda = 0.5; \rho = 1.6; \mu = 0.7; D_1 = 0.005; D_2 = 0.03; D_3 = 0.09;$

4. CONCLUSIONS

(i).This paper discussed rich dynamics of non-diffusion and diffusion variant of P-H-C model of food chain system.

(ii).The O.D.E version of the Holling Type-II and Type-IV P-H-C food chain model was proposed and discussed the boundedness of the solutions, global and local stability of the system at interior steady states.

(iii).The occurrence of Hpof bifurcation analysis was discussed and results showed in Fig:10-11

(iv).The diffusion version of the model proposed and discussed both the stability and instability (Turing stability) conditions for the proposed system at coexisting study state .From the numerical results, it is evident that the system (2.3) is said to be stable only, if the conditions of the theorem (3.2.1) are satisfied. The proposed system is said to be unstable, presented if the conditions of the theorem (3.2.2) are satisfied. It is shown that if all species undergo substantial self-diffusion, the equilibrium state is always stable and the Eigen values are suitable at large.

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