Mathematical Study of Plant-Herbivore-Carnivore System with Holling Type-II Functional Responses

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Abstract--- In this paper a systematic study is made, which is aimed at exploring a food chain model related to plants, the herbivore animals and carnivore animals and their impact on the global ecosystem. The study establishes the behavioural dynamics among the three participating species. The proposed model adopts a functional response traceable to Holling type-II for grazing of plants by herbivores and the hunting of the herbivores by the carnivores for their survival. The boundaries of the solution and Hopf bifurcation analysis are discussed at the positive steady state. The persistence (tendency of each specie to resist the invading specie), the global property of dynamic systems is discussed as a part of this study. The statistical randomness of population of each selected specie in the steady state of co-existence due to white noise is also computed. Finally, numerical illustrations are presented to support the results of the study.

Keywords-- Herbivores, Carnivores, Local Stability, Routh-Hurwitze Criteria, Dulac Criteria, Hopf Bifurcation, Persistence, Stochastic.

I. Introduction

The interrelation between completely different biological process levels in an ecological food-chain system has induced interest in mathematical ecologists for a long time. The study of the dynamics of this relationship is one of the dominant subjects in mathematical ecology which can be obtained through the formulation and analysis of mathematical models. Primary producers and primary consumers are the building blocks of biological process levels in an ecological food chain system. Plants (the primary producers) are capable of producing their food requirement from photosynthesis or by inorganic oxidation. The primary consumers are herbivores which prey on primary producers, the plants for their food source. Again the primary consumers, the herbivores as a most favorable food source for carnivores. Often, there are instances of carnivores becoming prey to species of carnivores. The seasonal environmental fluctuations have also a profound effect on these factors.

The food web models of those three species are elementary building blocks of huge scale ecosystem. The essential understanding of interactive dynamics of 3-species food cycle models are helpful to study the short term or long term behavior of ecosystems. The systems of differential equations can be used to model a food cycle which approximates species or the behavior of functional feeding group with various functional responses.

Many of the ecological models considered in the ecological literature [16, 18, 19, 8] are constructed by involving functional responses. Mukhopadhyay, B. Bhattacharya [9] studied an ecological food chain model nutrient-autotroph-herbivore model. In this model the Holling type-II functional response of herbivore is considered. Panja.P, Mondal. SK [11], Rangithkumar Upadhyay [13] studied a phytoplankton-Zooplankton and fish model having functional responses of any two species by third species is of Holling type-II.

In nature, the life history of members of many species may be separated into distinct stages. These stages represent the flexibility and the resistance within species offered to natural enemies at different stages of its growth. It will have an effect on the persistence and extinction of biological population to varying degrees. Meng etall [25] investigates the steadiness and Hopf bifurcation in a three-species system with stage structure for the predation. Hariyanto [23] mentioned the analysis of the persistence of the dynamical system.

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Many researchers observed that the environmental fluctuations also caused the different behaviors of the dynamic systems. J.Ripa [22] studied the effect of environmental noise in ecological food webs. R.M.May [21] investigated that the population has deviated more from steady states in a biological system involved in stochastic fluctuations by considering white noise for a population. Some of these studies motivated us to compute the population stochasticity around the steady state of co-existence due to white noise.

II. The Food Chain Model

In this paper, the plants act as resource biomass. The herbivores grow by eating plants and a carnivore grows by eating herbivores. In this model of Plants-Herbivores-Carnivores (P-H-C), plants are purely prey. Herbivores are food source for carnivores and simultaneously food source for herbivores is plants population. Here herbivore acts as both prey and predator simultaneously.

P(t), H(t) and C(t) denote the density of plants, herbivores and carnivores respectively at any instant of time t.

Assumptions

i. The parameters r and s are intrinsic growth rates of plants and herbivores respectively, and it is also assumed that the growth of the plants and herbivores is logistic.

ii. The parameters β_1 and γ_1 denote the attack rate at which the single herbivores searches for plant and the single carnivores searches for herbivores, whenever predator is not currently consuming prey item.

iii. The parameters α_1 and α_2 are half saturation levels of Herbivores and carnivores respectively.

iv. The parameters k and l are carrying capacities of plant populations and herbivores population respectively.

v. The parameter β_0 is the mortality rate of the carnivores.

vi. The term $F(P,H)=\dfrac{\beta_1 PH}{\alpha_1+P}$ represents the functional response for feeding of plants by herbivores. Here α_1 is the half saturation constant.

vii. The term $G(H,C)=\dfrac{\gamma_1 HC}{\alpha_2+H}$ indicates the functional response for feeding of herbivores by carnivores.

The P-H-C food chain model is mathematically represented as follows:

$$\begin{aligned}\frac{dP}{dt}&=rP\left(1-\frac{P}{k}\right)-\frac{\beta_1 PH}{\alpha_1+P}\\ \frac{dH}{dt}&=sH\left(1-\frac{H}{l}\right)+\frac{\beta_2 PH}{\alpha_1+P}-\frac{\gamma_1 HC}{\alpha_2+H}\\ \frac{dC}{dt}&=\frac{\gamma_2 HC}{\alpha_2+H}-\beta_0 C.\end{aligned}\tag{2.1}$$

The system (2.1) has thirteen parameters. It is evident that dealing a system having more number of parameters is challenging and required more complicated analysis. Reformulating a model in dimensionless type is helpful from many aspects. This procedure will facilitate to see the consistency of the model equations and ensure that each one terms have an equivalent set of units in equation. Additionally, non-dimensionalizing a model reduces the amount of free parameters and divulges a smaller set of quantities that govern the dynamics.

By non-dimensionalization, define

i) Non dimensional time: $t=t^{\odot}.\tau$

ii) Non dimensional plant density: $P=\overset{\odot}{P}.\overset{\oplus}{P}$

iii) Non dimensional herbivore density: $H=\overset{\odot}{H}.\overset{\oplus}{H}$

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iv) Non dimensional carnivore density: $C=\overset{\odot}{C}.\overset{\oplus}{C}$, where $\overset{\oplus}{P},\overset{\oplus}{H},\overset{\oplus}{C}$ and τ are time independent constants.

By substitution these expressions, system (2.1) becomes

$$\begin{aligned}
\frac{d\overset{\odot}{P}}{d\overset{\odot}{t}} &= r\overset{\odot}{P}\tau\left(1-\frac{\overset{\odot}{P}.\overset{\oplus}{P}}{k}\right)-\frac{\beta_1\overset{\odot}{P}\overset{\odot}{H}\overset{\oplus}{H}\tau}{\alpha_1+\overset{\odot}{P}.\overset{\oplus}{P}}\\
\frac{d\overset{\odot}{H}}{d\overset{\odot}{t}} &= s\overset{\odot}{H}\tau\left(1-\frac{(\overset{\odot}{H}.\overset{\oplus}{H})}{l}\right)+\frac{\beta_2\overset{\odot}{P}\overset{\oplus}{P}\overset{\odot}{H}\tau}{\alpha_1+\overset{\odot}{P}.\overset{\oplus}{P}}-\frac{\gamma_1\overset{\odot}{H}\overset{\odot}{C}\overset{\oplus}{C}\tau}{\alpha_2+\overset{\odot}{H}.\overset{\oplus}{H}}\\
\frac{d\overset{\odot}{C}}{d\overset{\odot}{t}} &= \frac{\gamma_2\overset{\odot}{H}\overset{\oplus}{H}\overset{\odot}{C}\tau}{\alpha_2+\overset{\odot}{H}.\overset{\oplus}{H}}-\beta_0\overset{\odot}{C}\tau.
\end{aligned}$$

By choosing $\overset{\oplus}{P}=\frac{k}{r\tau},\overset{\oplus}{H}=\frac{l}{s\tau},\overset{\oplus}{C}=\frac{1}{\gamma_1\tau},\tau=\frac{1}{\beta_0}$, the non-dimensional form of system (2.1) is

$$\begin{aligned}
\frac{dP}{dt} &= \alpha P-P^2-\frac{\beta PH}{\gamma+P}=Pf_1(P,H,C)\\
\frac{dH}{dt} &= \delta H-H^2+\frac{\lambda PH}{\gamma+P}-\frac{HC}{\mu+H}=\mathrm{H}f_2(P,H,C)\\
\frac{dC}{dt} &= \frac{\rho HC}{\mu+H}-C=Cf_3(P,H,C).
\end{aligned} \tag{2.2}$$

Where $\alpha=\frac{r}{\beta_0},\beta=\frac{\beta_1 l}{s},\gamma=\frac{\alpha_1 r}{\beta_0 k},\delta=\frac{s}{\beta_0},\lambda=\frac{\beta_2 k}{r},\mu=\frac{\alpha_2 s}{\beta_0 l},\rho=\frac{\gamma_2 l}{s}.$

III. Existence and Boundedness of the System

The right-hand side of system (2.2) is continuous and has continuous partial derivatives on the state space $R_+^3=\left\{(P,H,C)\in R^3: P\geq 0, H\geq 0, C\geq 0\right\}$. Therefore, the solution of the system (2.2) with non-negative initial condition exists; it is unique, and uniformly bounded.

Theorem (3.1) *when* $\eta\leq\left\{\min(\alpha,\delta)\right\}$ *and* $\beta>\lambda$, $\rho<1$, *the solutions of system (2.2) are uniformly bounded for the positive parameter* η.

Proof: - consider (P,H,C) be the solution of the system (2.2) with positive initial condition such that

$$W=P+H+C\text{, then }\frac{dW}{dt}=\frac{dP}{dt}+\frac{dH}{dt}+\frac{dC}{dt}.$$

$$\frac{dW}{dt}=(\alpha P-P^2+\delta H-H^2-C)-\frac{PH}{\gamma+P}(\beta-\lambda)-\frac{HC}{\mu+H}(1-\rho)$$

Assume that $\beta>\lambda$ and $\rho<1$, then

$$\frac{dW}{dt}\leq(\alpha P-P^2+\delta H-H^2-C)\leq(2\alpha P-P^2+2\delta H-H^2)$$

When $\eta=\left\{\min(\alpha,\delta)\right\}$, then

$$\frac{dW}{dt}+\eta W\leq(\alpha^2-(P-\alpha)^2+\delta^2-(H-\delta)^2\leq\alpha^2+\delta^2=\upsilon$$

$$\frac{dW}{dt}+\eta W\leq\upsilon\Rightarrow W=\frac{\upsilon}{\eta}+me^{-\eta t}\text{, where }m=W(0)-\frac{\upsilon}{\eta}$$

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$$\Rightarrow W\left((P,H,C\right) \leq \frac{\upsilon}{\eta} + \left(W(0) - \frac{\upsilon}{\eta}\right)e^{-\eta t} \leq \frac{\upsilon}{\eta}\left(1 - e^{-\eta t}\right) + W(0)e^{-\eta t}$$

As $t \to \infty$, $W\left(t\right)$ lies between 0 and $\frac{\upsilon}{\eta}$, provided $\beta > \lambda$ and $\rho < 1$.

Therefore, the region $\Omega = \left\{\left(P,H,C\right) \in R_{+}^{3} : W = P + H + c \leq \frac{\upsilon}{\eta}\right\}$ in which the solutions of the system (2.2) are uniformly bounded. □

IV. Steady States

The system has the following six steady state solutions resulting from $\frac{dP}{dt} = 0, \frac{dH}{dt} = 0, \frac{dC}{dt} = 0$.

a) The trivial steady state $E_0 = (0,0,0)$.

b) The steady state $E_1 = (0,\delta,0)$.

c) The steady state $E_2 = (\alpha,0,0)$.

d) The boundary steady state $E_3 = (0,\bar{H},\bar{C})$ on the H-C plane

Where $\bar{H} = \frac{\mu\rho}{\rho-1}, (\rho > 1)$ and $\bar{C} = \frac{\mu\rho}{\rho-1}\left(\delta - \frac{\mu}{\rho-1}\right)$.

e) The planar steady state $E_4 = (\hat{P},\hat{H},0)$ on the P-H plane, where $\hat{H} = \delta + \frac{\lambda\hat{P}}{\gamma+\hat{P}}$ and $\hat{P} = \frac{1}{2}\left[(\alpha-\gamma) \pm \sqrt{(\alpha-\gamma)^2 - 4\left(\beta\hat{H} - \alpha\gamma\right)}\right]$

f) The positive steady state $E_5 = (\overset{*}{P},\overset{*}{H},\overset{*}{C})$, where $\overset{*}{H} = \frac{\mu}{\rho-1}, (\rho > 1)$ $\overset{*}{P} = \frac{1}{2}\left[(\alpha-\gamma) \pm \sqrt{(\alpha-\gamma)^2 - 4\left(\frac{\beta\mu - \alpha\gamma(\rho-1)}{\rho-1}\right)}\right]$ and $\overset{*}{C} = \frac{\mu}{(\rho-1)^2}\left[\delta\rho + \frac{(\rho-1)\lambda\overset{*}{P}}{\gamma+\overset{*}{P}} - (\delta+\mu)\right]$.

V. Existence and Stability Analysis of Steady States

Clearly, the three steady states E_0, E_1 and E_3 always exist and stable.

Theorem (5.1): If $1 < \rho < \frac{\alpha\gamma(\delta+\mu)}{\alpha\gamma(\delta+\mu) - 2\mu\beta\delta}$, the boundary steady state $E_3 = (0,\bar{H},\bar{C})$ exists and it is stable.

Proof:-For the point $E_3 = (0,\bar{H},\bar{C})$ the corresponding Jacobian Matrix is

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$$J\left(E_3\right)=\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & 0 \end{pmatrix},$$

Where

$$a_{11}=\alpha-\frac{\beta\mu}{\gamma\left(\rho-1\right)},a_{21}=\frac{\lambda\mu}{\gamma\left(\rho-1\right)},a_{22}=\frac{\delta+\mu}{\rho}-\frac{2\mu}{\rho-1},a_{23}=-\frac{1}{\rho},a_{32}=\delta\left(\rho-1\right)-\mu$$

The characteristic equation of $J\left(E_3\right)$ is $\left(a_{11}-\lambda\right)\left[\lambda^2-a_{22}\lambda-a_{23}a_{32}\right]=0.$ If $a_{11}<0,a_{22}<0$ and $a_{23}a_{32}<0$, which implies that $1<\rho<\frac{\alpha\gamma(\delta+\mu)}{\alpha\gamma(\delta+\mu)-2\mu\beta\delta}$.The steady state $E_3=(0,\bar{H},\bar{C})$ is stable, otherwise it is unstable.

Theorem (5.2) Along with the condition stated in theorem (5.1), the steady state $E_3=(0,\bar{H},\bar{C})$ is globally stable in the H-C plane if $\mu=\delta$

Proof: For any initial value in the H-C plane, the system (2.2) becomes

$$\frac{dH}{dt}=\delta H-H^2-\frac{HC}{\mu+H}=g_1\left(H,C\right) \quad \text{And} \quad \frac{dC}{dt}=\frac{HC\rho}{\mu+H}-C=g_2\left(H,C\right).$$

Assume that $M(H,C)=\frac{\mu+H}{C}$, clearly $M(H,C)>0\forall(H,C)\in$ interior of R_+^2.

Now

$$\nabla.\left[M(H,C)\right].\begin{bmatrix}\frac{dH}{dt}\\ \frac{dC}{dt}\end{bmatrix}=\nabla.\left[\left(\frac{\mu+H}{C}\right)\begin{pmatrix}g_1\left(H,C\right)\\ g_2\left(H,C\right)\end{pmatrix}\right]==\frac{\delta}{C}-\frac{1}{C}\left(\mu+2H\right)=-\frac{1}{C}\left(2H+\left(\mu-\delta\right)\right)$$

When $\mu=\delta$, this expression reduces to $-\frac{1}{C}\left(2H\right)<0$, which is negative everywhere .Then according to Bendixson-Dulac theorem, the periodic solutions does not exists in H-C plane. Since, all the solutions of the system are bounded and E_3 are unique positive steady state in H-C plane; hence by Poincare Bendixson-Dulac theorem the steady state $E_3=(0,\bar{H},\bar{C})$ is globally stable.

Theorem (5.3) *Assume that the condition in the theorem (5.1) holds and if* $\lambda<\frac{\left(\gamma+P\right)\left(H-\bar{H}\right)}{P}$ *then the steady state* E_3 *is globally stable.*

Proof: Consider the function

$$V_1\left(H,C\right)=l_1\left[H-\bar{H}-\bar{H}\ln\left(\frac{H}{\bar{H}}\right)\right]+l_2\left[C-\bar{C}-\bar{C}\ln\left(\frac{C}{\bar{C}}\right)\right],$$

which is positive definite and l_1, l_2 are positive constants to be determined.

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$$\frac{dV_1}{dt} = l_1\left[\frac{H-\bar{H}}{H}\right].\frac{dH}{dt} + l_2\left[\frac{C-\bar{C}}{C}\right].\frac{dC}{dt}$$

$$= -l_1\left(H-\bar{H}\right)^2 - l_1\frac{\left(H-\bar{H}\right)\left(C-\bar{C}\right)}{\bar{H}+\mu} + l_2\frac{\mu\left(H-\bar{H}\right)\left(C-\bar{C}\right)}{\bar{H}+\mu} + l_1\frac{P\lambda\left(H-\bar{H}\right)}{\gamma+P}$$

By choosing non-negative constants $l_1 = \left(\frac{\mu\left(\bar{H}+\mu\right)}{\bar{H}\left(\mu+H\right)}\right)$, $l_2 = 1$, and if $\lambda < \frac{\left(\gamma+P\right)\left(H-\bar{H}\right)}{P}$

$\frac{dV_1}{dt} < 0$, Therefore by Lyapunov theorem the steady state $E_3 = (0,\bar{H},\bar{C})$ is globally stable.

Theorem (5.4) The steady state $E_4 = (\hat{P},\hat{H},0)$ exists, if $2\gamma < \alpha, \beta\left(\delta+\lambda\right)+\gamma^2 < 2\alpha\gamma$ and $\beta\delta < \alpha\gamma$.

Proof: Let $\hat{P},\hat{H}$ are the solutions of the equations: $\alpha - \hat{P} - \frac{\hat{H}\beta}{\gamma+\hat{P}} = 0$ and

$\delta - \hat{H} + \frac{\hat{P}\lambda}{\gamma+\hat{P}} - \frac{\hat{C}}{\mu+\hat{H}} = 0$ .By solving first equation $\hat{H} = \delta + \frac{\hat{P}\lambda}{\gamma+\hat{P}}$ and by substituting it in the second equation we get $\hat{P}^3 - \left(\alpha-2\gamma\right)\hat{P}^2 - \left[2\alpha\gamma - \beta\left(\delta+\lambda\right) - \gamma^2\right]\hat{P} - \gamma\left(\alpha\gamma - \beta\delta\right) = 0$ .By Descartes' rule of sign, the positive solution $\hat{P}$ exists, if $2\gamma < \alpha, \beta\left(\delta+\lambda\right)+\gamma^2 < 2\alpha\gamma$ and $\beta\delta < \alpha\gamma$ .Therefore, the positive solutions $\hat{H} = \delta + \frac{\hat{P}\lambda}{\gamma+\hat{P}}$ and $\hat{P} = \frac{1}{2}\left[\left(\alpha-\gamma\right) \pm \sqrt{\left(\alpha-\gamma\right)^2 - 4\left(\beta\hat{H} - \alpha\gamma\right)}\right]$ are exists,

if $2\gamma < \alpha, \beta\left(\delta+\lambda\right)+\gamma^2 < 2\alpha\gamma$, $\beta\delta < \alpha\gamma$

Theorem (5.5): The steady state $E_4 = (\hat{P},\hat{H},0)$ is locally asymptotically stable,

if $c_{33} < 0, c_{11}+c_{22} < 0$ and $c_{11}c_{22} - c_{12}c_{21} > 0$ otherwise, it is unstable.

Proof: For the point $E_4 = (\hat{P},\hat{H},0)$ the corresponding variation matrix is

$$J\left(E_4\right) = \begin{pmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & c_{23} \\ 0 & 0 & c_{33} \end{pmatrix}$$

Where

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$$c_{11} = \frac{\hat{H}\beta\gamma}{\left(\gamma+\hat{P}\right)^2} - \hat{P}, c_{12} = \frac{-\beta\hat{P}}{\left(\gamma+\hat{P}\right)}, c_{21} = \frac{\hat{H}\alpha\gamma}{\left(\gamma+\hat{P}\right)^2}, c_{22} = -\hat{H}, c_{23} = \frac{-\hat{H}}{\mu+\hat{H}}, c_{33} = \frac{\hat{H}\rho}{\mu+\hat{H}} - 1$$

The characteristic equation of $J(E_4)$ is $(c_{33}-\lambda)\left[\lambda^2-(c_{11}+c_{22})\lambda+(c_{11}c_{22}-c_{12}c_{21})\right]=0$ and the Eigen values are $\lambda_1 = c_{33}, \lambda_2+\lambda_3 = c_{11}+c_{22}, \lambda_2.\lambda_3 = c_{11}c_{22}-c_{12}c_{21}$ .If $c_{33}<0, c_{11}+c_{22}<0$ and $c_{11}c_{22}-c_{12}c_{21}>0$, then the steady state $E_4 = (\hat{P},\hat{H},0)$ is locally asymptotically stable otherwise, it is unstable.

Theorem (5.6) Along with the conditions stated in theorems (5.4) and (5.5), and if $\gamma=\alpha$

The steady state $E_4 = (\hat{P},\hat{H},0)$ is globally stable in the P-H plane.

Proof: For any initial value in the P- H plane, the system (2.2) becomes

$$\frac{dP}{dt} = \alpha P - P^2 - \frac{PH\beta}{\gamma+P} = g_3(P,H) \text{ and } \frac{dH}{dt} = \delta H - H^2 + \frac{PH\lambda}{\gamma+P} = g_4(P,H)$$

Let $M(P,H) = \dfrac{\gamma+P}{PH}$, clearly $M(P,H)>0 \forall (P,H) \in$ interior R_+^2.

Now

$$\nabla.[M(P,H)].\begin{bmatrix}\frac{dP}{dt}\\ \frac{dH}{dt}\end{bmatrix} = \nabla.\left[\left(\frac{\gamma+P}{PH}\right)\begin{pmatrix}g_3(P,H)\\ g_4(P,H)\end{pmatrix}\right] = \frac{-1}{H}\left[2P+(\gamma-\alpha)\right]-\left(\frac{\gamma+P}{P}\right),$$ when $\gamma=\alpha$, this expression reduces to $\dfrac{-1}{H}[2P]-\left(\dfrac{\gamma+P}{P}\right)<0$ which is negative everywhere. Then according to Bendixson-Dulac theorem, the periodic solution does not exists in P-H plane. Since, all the solutions of the system are bounded and E_3 are unique positive steady state in P-H plane; hence by Poincare Bendixson-Dulac theorem the steady state $E_4 = (\hat{P},\hat{H},0)$ is globally stable.

Theorem (5.7) Assume that the condition in the theorem (5.4) holds and if $\beta H < (\gamma+P)\left(\gamma+\hat{P}\right)$ the planar steady state $E_4 = (\hat{P},\hat{H},0)$ is globally asymptotically stable.

Proof:-Consider the function

$$V_2(P,H) = m_1\left[P-\hat{P}-\hat{P}\ln\left(\frac{P}{\hat{P}}\right)\right]+m_2\left[H-\hat{H}-\hat{H}\ln\left(\frac{H}{\hat{H}}\right)\right]$$, where m_1 and m_2 are positive constants to be determined.

$$\frac{dV_2}{dt} = m_1\left[\frac{P-\hat{P}}{P}\right].\frac{dP}{dt}+m_2\left[\frac{H-\hat{H}}{H}\right].\frac{dH}{dt}$$

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$$\frac{dV_2}{dt}=-m_1\left[P-\hat{P}\right]^2-m_2\left[H-\hat{H}\right]^2-m_1\beta\gamma\frac{\left(P-\hat{P}\right)\left(H-\hat{H}\right)}{(\gamma+P)\left(\gamma+\hat{P}\right)}-m_1P\beta\frac{\left(P-\hat{P}\right)\left(H-\hat{H}\right)}{(\gamma+P)\left(\gamma+\hat{P}\right)}+m_1\frac{\left(P-\hat{P}\right)^2\beta H}{(\gamma+P)\left(\gamma+\hat{P}\right)}+$$

$$m_2\lambda\gamma\frac{\left(P-\hat{P}\right)\left(H-\hat{H}\right)}{(\gamma+P)\left(\gamma+\hat{P}\right)}-m_2C\frac{\left(H-\hat{H}\right)}{\mu+H}$$

By Choosing non-negative constants $m_1=1$, $m_2=\left(\frac{\beta(\gamma+P)}{\lambda\gamma}\right)$ and If $\beta H<(\gamma+P)\left(\gamma+\hat{P}\right)$,

$\frac{dV_2}{dt}<0$ .Therefore, by Lyapunov theorem the steady state $E_4=(\hat{P},\hat{H},0)$ is globally asymptotically stable.

Theorem (5.8): The *interior equilibrium point* $E_5=(\overset{*}{P},\overset{*}{H},\overset{*}{C})$ *exists, if*

$$\rho>1,(\alpha-\gamma)^2>4\left(\frac{\beta\mu}{\rho-1}-\alpha\gamma\right)\text{ and }\left[\delta\rho+\frac{(\rho-1)\overset{*}{P}\lambda}{\gamma+\overset{*}{P}}\right]>(\delta+\mu).$$

Proof:-Let $\overset{*}{P},\overset{*}{H},\overset{*}{C}$ are positive solutions of the equations $\alpha-\overset{*}{P}-\frac{\overset{*}{H}\beta}{\gamma+\overset{*}{P}}=0,\ \delta-\overset{*}{P}+\frac{\overset{*}{P}\lambda}{\gamma+\overset{*}{P}}-\frac{\overset{*}{C}}{\mu+\overset{*}{H}}=0$

and $\frac{\overset{*}{H}\rho}{\mu+\overset{*}{H}}-1=0$. By solving these equations we obtain

$$\overset{*}{H}=\frac{\mu}{\rho-1},\ \overset{*}{P}=\frac{1}{2}\left[(\alpha-\gamma)\pm\sqrt{(\alpha-\gamma)^2-4\left(\frac{\beta\mu-\alpha\gamma(\rho-1)}{\rho-1}\right)}\right]\text{and}\ \overset{*}{C}=\frac{\mu}{(\rho-1)^2}\left[\delta\rho+\frac{(\rho-1)\lambda\overset{*}{P}}{\gamma+\overset{*}{P}}-(\delta+\mu)\right].$$

Hence, the interior steady state $E_5=(\overset{*}{P},\overset{*}{H},\overset{*}{C})$ exists, if $(\alpha-\gamma)^2>4\left(\frac{\beta\mu}{\rho-1}-\alpha\gamma\right)$ *and* $(\rho>1)$, $\left[\delta\rho+\frac{(\rho-1)\overset{*}{P}\lambda}{\gamma+\overset{*}{P}}\right]>(\delta+\mu)$.

Theorem **(5.9):** The interior steady state $E_5=(\overset{*}{P},\overset{*}{H},\overset{*}{C})$ is locally asymptotically stable, if $B_1>0, B_3>0$ and $(B_1B_2-B_3)>0$, otherwise, it is unstable.

Proof:-For the point $E_5=(\overset{*}{P},\overset{*}{H},\overset{*}{C})$ the corresponding Jacobian matrix is

$$J(E_5)=\begin{pmatrix}b_{11}&b_{12}&0\\b_{21}&b_{22}&b_{23}\\0&b_{32}&0\end{pmatrix},$$

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Where

$$b_{11}=\frac{\overset{*}{H}\,\overset{*}{P}\beta}{\gamma+\overset{*}{P}}-\overset{*}{P},b_{12}=\frac{-\overset{*}{P}\beta}{\gamma+\overset{*}{P}},b_{21}=\frac{\overset{*}{H}\lambda\gamma}{\left(\gamma+\overset{*}{P}\right)^2},b_{22}=\frac{\overset{*}{C}\,\overset{*}{H}}{\left(\mu+\overset{*}{H}\right)^2}-\overset{*}{H},b_{23}=-\frac{\overset{*}{H}}{\mu+\overset{*}{H}},b_{32}=\frac{\overset{*}{C}\mu\rho}{\left(\mu+\overset{*}{H}\right)^2}.$$

The characteristic equation of $J\left(E_5\right)$ is $\lambda^3+B_1\lambda^2+B_2\lambda+B_3=0$.

Here $B_1=-\left(b_{11}+b_{22}\right), B_2=\left(b_{11}b_{22}-b_{32}b_{23}-b_{12}b_{21}\right), B_3=b_{11}b_{32}b_{23}.$

Now

$B_1=\frac{\overset{*}{H}}{N_2^2N_3^2}\left[N_1+\frac{\overset{*}{P}}{\overset{*}{H}}N_2^2N_3^2\right]; B_3=\frac{\overset{*}{P}\overset{*}{C}\mu^2\rho}{N_2^3}\left(\frac{N_3^2-\overset{*}{H}\beta}{N_3^2}\right)$; and N_1, N_2 are defined by

$N_1=N_2^2N_3^2-\left(\overset{*}{C}N_3^2+\overset{*}{P}\beta N_2^2\right), N_2=\left(\mu+\overset{*}{H}\right)>0$, $N_3=\left(\gamma+\overset{*}{P}\right)>0$. If $\beta<\frac{N_2^2N_3^2}{\overset{*}{C}\overset{*}{H}+\overset{*}{P}N_2^2}$, the coefficients of characteristic equation $B_1>0$ and $B_3>0$. Again,

consider $\Delta=B_1B_2-B_3=\left(b_{11}+b_{22}\right)\left(b_{12}b_{21}-b_{11}b_{22}\right)+\left(b_{22}b_{32}b_{23}\right)$.

$$\Delta=\frac{\overset{*}{P}\overset{*}{H}}{N_2^2N_3^2}\left\{\left[N_1\overset{*}{H}+\overset{*}{P}N_2^2N_3^2\right].\left[\left(\overset{*}{H}\beta-N_3^2\right)\left(\overset{*}{C}-N_2^2\right)+\frac{\lambda\beta\gamma}{N_3^3}\right]+\frac{\overset{*}{C}\mu^2\rho N_3^2\left(N_2^2-\overset{*}{C}\right)}{\overset{*}{P}N_2^3}\right\}.$$

$\Rightarrow\Delta=B_1B_2-B_3>0$ if $\lambda>\frac{\left(\overset{*}{C}-N_2^2\right)N_3^3}{\beta\gamma}\left[\left(N_3^2-\overset{*}{H}\beta\right)+\frac{\overset{*}{C}\mu^2\rho N_3^2}{\overset{*}{P}N_2^3\left(N_1\overset{*}{H}+\overset{*}{P}N_2^2N_3^2\right)}\right]$. Therefore

By Routh-Hurwitze criteria, the interior steady state $E_5=(\overset{*}{P},\overset{*}{H},\overset{*}{C})$ is locally asymptotically stable, if. $B_1>0, B_3>0$ and $\left(B_1B_2-B_3\right)>0$.

Theorem (5.10) Along with the conditions stated in the theorems (5.7), (5.8) and **if** $\beta H<\left(\gamma+P\right)\left(\gamma+\overset{*}{P}\right)$ the steady state $E_5=(\overset{*}{P},\overset{*}{H},\overset{*}{C})$ is globally asymptotically stable.

Proof: Consider the positive definite function

$$V_3\left(P,H,C\right)=n_1\left[P-\overset{*}{P}-\overset{*}{P}\ln\left(\frac{P}{\overset{*}{P}}\right)\right]+n_2\left[H-\overset{*}{H}-\overset{*}{H}\ln\left(\frac{H}{\overset{*}{H}}\right)\right]+n_3\left[C-\overset{*}{C}-\overset{*}{C}\ln\left(\frac{C}{\overset{*}{C}}\right)\right],$$

where n_1,n_2 and n_3 are positive constants to be determined.

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$$\frac{dV_3}{dt}=n_1\left[\frac{P-\overset{*}{P}}{P}\right].\frac{dP}{dt}+n_2\left[\frac{H-\overset{*}{H}}{H}\right].\frac{dH}{dt}+n_3\left[\frac{C-\overset{*}{C}}{C}\right].\frac{dC}{dt}$$

$$\frac{dV_3}{dt}=-n_1\left[P-\overset{*}{P}\right]^2-n_2\left[H-\overset{*}{H}\right]^2-n_1\beta\gamma\frac{\left(P-\overset{*}{P}\right)\left(H-\overset{*}{H}\right)}{(\gamma+P)\left(\gamma+\overset{*}{P}\right)}-n_1P\beta\frac{\left(P-\overset{*}{P}\right)\left(H-\overset{*}{H}\right)}{(\gamma+P)\left(\gamma+\overset{*}{P}\right)}+n_1\frac{\left(P-\overset{*}{P}\right)^2\beta H}{(\gamma+P)\left(\gamma+\overset{*}{P}\right)}+$$

$$n_3\left[C-\overset{*}{C}\right].\left(\frac{H\rho}{\mu+H}-1\right)+n_2\lambda\gamma\frac{\left(P-\overset{*}{P}\right)\left(H-\overset{*}{H}\right)}{(\gamma+P)\left(\gamma+\overset{*}{P}\right)}-n_2C\frac{\left(H-\overset{*}{H}\right)}{\mu+H}.$$

Chose non-negative constants $n_1=1, n_2=\left(\frac{\beta(\gamma+P)}{\lambda\gamma}\right)$ $n_3=\left(\frac{C\overset{*}{H}n_2}{\left(C-\overset{*}{C}\right)\mu}\right)$, and if $\beta H<(\gamma+P)\left(\gamma+\overset{*}{P}\right)$

, $\frac{dV_3}{dt}<0$. Therefore by Lyapunov theorem the steady state $E_5=(\overset{*}{P},\overset{*}{H},\overset{*}{C})$ is globally asymptotically stable.

VI. Hopbifurcation

In the present study, various parameters have been used to exhibit the behavior of dynamical system. PHC models with constant parameters are frequently found to approach a steady state where species coexist in equilibrium. The behavior of a system may change in relation to the parameters used in the model. Such parameters which cause the transition in a system are named as bifurcation points. At any point where the system has nontrivial periodic solutions, a Hopf bifurcation occurs.

The following theorem established that Hopfbifurcation occurs for the system (2.2) at a critical value $\lambda=\overset{*}{\lambda}$.For proving this, we follow Liu [7] approach.

Theorem (6.1) *Assume that* $\beta<\frac{N_2^2N_3^2}{\overset{*}{C}\overset{*}{H}+\overset{*}{P}N_2^2}$ *holds, then a simple Hopfbifurcation of the system (2.2) occurs at* $\lambda=\overset{*}{\lambda}$.

Proof. *Assume* that the local stability conditions hold, and

let $\overset{*}{\lambda}=\frac{\left(\overset{*}{C}-N_2^2\right)N_3^3}{\beta\gamma}\left[\left(N_3^2-\overset{*}{H}\beta\right)+\frac{\overset{*}{C}\mu^2\rho N_3^2}{\overset{*}{P}N_2^3\left(N_1\overset{*}{H}+\overset{*}{P}N_2^2N_3^2\right)}\right]$, then

$$B_1\Big|_{\lambda=\overset{*}{\lambda}}=\frac{\overset{*}{H}}{N_2^2N_3^2}\left[N_1+\frac{\overset{*}{P}}{\overset{*}{H}}N_2^2N_3^2\right]>0,\quad B_3\Big|_{\lambda=\overset{*}{\lambda}}=\frac{\overset{*}{P}\overset{*}{C}\mu^2\rho}{N_2{}^3}\left(\frac{N_3{}^2-\overset{*}{H}\beta}{N_3{}^2}\right)>0$$

And $\frac{d\Delta}{d\lambda}\Big|_{\lambda=\overset{*}{\lambda}}=\frac{\overset{*}{P}\overset{*}{H}\beta\gamma}{N_2^2N_3^5}\left[N_1\overset{*}{H}+\overset{*}{P}N_2^2N_3^2\right]\neq 0$

Therefore, $\frac{d\Delta}{d\lambda}\Big|_{\lambda=\overset{*}{\lambda}}\neq 0$.Hence, a simple Hopf bifurcation occurs at $\lambda=\overset{*}{\lambda}$. The graphical presentation has been shown in Fig. 6 and Fig. 8.

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VII. Persistence Analysis

In this section, the persistence of system (2.2) is studied. When each and every species inside a system is persistent, the system as a whole is said to be persistent. As per the approach made by Freedman and Waltman [20], the system is said to be persistence when the solution of the system does not possess Omega Limit Set with non-negative initial condition on its boundary planes.

Theorem (7.1) Assume that the steady state $E_4 = (\hat{P}, \hat{H}, 0)$ is a globally stable in the PH-plane. Then the necessary and sufficient conditions for the persistence of the system (2.2) are

$$\lambda_1 = \left(\frac{\hat{H} \rho}{\mu + \hat{H}} - 1 \right) \geq 0 \text{ and } \lambda_1 = \left(\frac{\hat{H} \rho}{\mu + \hat{H}} - 1 \right) > 0 \qquad (7.1)$$

Proof: The boundedness of the solution and the global stability of the system (2.2) are proved in theorems (3.1) and (5.7) respectively. Also, whenever the steady state E_4 exists, the Eigen value λ_1 gives the stability in the positive direction orthogonal to PH- plane. If $\lambda_1 < 0$, there are orbits in the positive cone approach steady state E_4.

Therefore $\hat{H} \rho \geq \mu + \hat{H}$ is the necessary condition for the persistence of the system (2.2).

For the sufficient condition $\hat{H} \rho > \mu + \hat{H}$, by Freedman and Walt man theorem [5, 20], the growth functions $f_{1,} f_{2,} f_3$ of the system (2.2) satisfies the following hypothesis

i)
$$\frac{\partial f_1}{\partial H} = \frac{-\beta}{\gamma + P} < 0, \frac{\partial f_1}{\partial C} = 0$$
$$\frac{\partial f_2}{\partial P} = \frac{\lambda}{\gamma + P} > 0, \frac{\partial f_2}{\partial C} = \frac{-1}{\mu + H} < 0$$
$$\frac{\partial f_3}{\partial P} = 0, \frac{\partial f_3}{\partial H} = \frac{\rho}{\mu + H} > 0$$

ii) In the absence of predator, the prey population grows to carrying capacity. That is $f_1(0,0,0) = \alpha > 0$, $f_2(0,0,0) = \delta > 0$ and $\frac{\partial f_1}{\partial P}(P,0,0) = -1 < 0,\ \frac{\partial f_2}{\partial H}(0,H,0) = -1 < 0$.

Further, the predator extinct (i.e. $f_3(0,0,0) = -1 < 0$) in the absence of the prey,

iii) The parameters β, λ are positive. Therefore, there are no steady states HC and PC planes.

iv) When the carnivores are not present, the herbivores can survive on its prey. Therefore there exists steady state $E_4 = (\hat{P}, \hat{H}, 0)$ in the PH-plane, such that $f_1(\hat{P}, \hat{H}, 0) = f_2(\hat{P}, \hat{H}, 0) = 0.$ hence, the system (2.2) persists if the condition (7.2) is satisfied.

Theorem (7.2) for each limit cycle $(\varphi_1(t), \varphi_2(t))$ the condition for persistence of the system is in the form $\int_0^T f_3(\varphi_1, \varphi_2, 0)\, dt > 0$, (7.2)

Where T is the time period of the limit cycle, such that the condition (7.2) holds and the PH-plane consists of a finite number of limit cycles.

Proof: - When the existence of a limit cycle in PH-plane is assumed, the Jacobean matrix about the cycle, $P = \varphi_1, H = \varphi_2, C = 0$ is

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$$V\left(\varphi_1(t),\varphi_2(t),0\right)=\begin{bmatrix}\varphi_1\dfrac{\partial f_1}{\partial P}+f_1 & \varphi_1\dfrac{\partial f_1}{\partial H} & 0\\ \varphi_2\dfrac{\partial f_2}{\partial P} & \varphi_2\dfrac{\partial f_2}{\partial H} & \varphi_2\dfrac{\partial f_2}{\partial C}\\ 0 & 0 & f_3\end{bmatrix} \qquad (7.3)$$

Where the partial derivatives and $f_i\ (i=1,2,3)$ in (7.3) are determined at $\left(\varphi_1,\varphi_2,0\right)$.

With the initial condition (t,p_0,h_0,c_0), consider the solution of the system (2.2) exists and which is close to $\left(\varphi_1,\varphi_2,0\right)$.

From the variation matrix (7.4), a solution of the system $\dfrac{\partial C}{\partial t}=f_3\left(\varphi_1(t),\varphi_2(t),0\right).C$ with $C(0)=1$ is $\dfrac{\partial C}{\partial c_0}$ Thus $\dfrac{\partial C}{\partial c_0}(t,p_0,h_0,c_0)=e^{\int_0^t f_3(\varphi_1(s),\varphi_2(s),0)ds}$.

Now, by applying Taylor's expansion, we have $C(t,p_0,h_0,c_0)-C(t,p_0,h_0,0)\cong c_0.e^{\int_0^t f_3(\varphi_1(s),\varphi_2(s),0)ds}$. Then C decrease or increase according to $\int_0^T f_3\left(\varphi_1,\varphi_2,0\right)dt$ is negative or positive respectively. The solution curves are away from the PH-plane if conditions (7.2) and (7.3) hold, since the PH-plane consists of a finite number of limit cycles.

VIII. Stochastic Analysis

In this section the stochastic version of the model has been formulated to take the influence of the random noise which in the form of additive Gaussian white noise to the model (2.2).The model is

$$\frac{dP}{dt}=\alpha P-P^2-\frac{\beta H}{\gamma+P}P+k_1\xi(t) \qquad (8.1)$$

$$\frac{dH}{dt}=\delta H-H^2+\frac{\lambda P}{\gamma+P}H-\frac{HC}{\mu+H}+k_2\xi_2(t) \qquad (8.2)$$

$$\frac{dC}{dt}=\frac{\rho HC}{\mu+H}-C+k_3\xi_3(t) \qquad (8.3)$$

Where P represents plants, H represents herbivores, C represents carnivores, $k_{1,}k_2,k_3$ are real constants $\xi=\left(\xi_1,\xi_2,\xi_3\right)$ is a 3D Gaussian White noise process satisfying $E\left[\xi_i\right]=0;i=1,2,3$

$E\left[\xi_i(t)\xi_j(t^{'})\right]=\delta_{ij}\delta\left(t-t^{'}\right);i=j=1,2,3$ Where δ_{ij}, δ are the Kronecker symbol and δ -Dirac function respectively.

Let $P=\eta_1+\overset{*}{S};H=\eta_2+\overset{*}{R};C=\eta_3+\overset{*}{T};$

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$$\frac{dP}{dt}=\dot{\eta}_1;\frac{dH}{dt}=\dot{\eta}_2;\frac{dC}{dt}=\dot{\eta}_3;$$

$$\dot{\eta}_1=\alpha(\eta_1+\overset{*}{S})-(\eta_1+\overset{*}{S})^2-\frac{\beta\left(\eta_2+\overset{*}{R}\right)\left(\eta_1+\overset{*}{S}\right)}{\gamma+\left(\eta_1+\overset{*}{S}\right)}+k_1\xi_1(t)$$

$$\dot{\eta}_2=\delta\left(\eta_2+\overset{*}{R}\right)-\left(\eta_2+\overset{*}{R}\right)^2+\frac{\lambda\left(\eta_1+\overset{*}{S}\right)\left(\eta_2+\overset{*}{R}\right)}{\gamma+\left(\eta_1+\overset{*}{S}\right)}-\frac{\left(\eta_2+\overset{*}{R}\right)\left(\eta_3+\overset{*}{T}\right)}{\mu+\left(\eta_2+\overset{*}{R}\right)}+k_2\xi_2(t)$$

$$\dot{\eta}_3=\rho\frac{\left(\eta_2+\overset{*}{R}\right)\left(\eta_3+\overset{*}{T}\right)}{\mu+\left(\eta_2+\overset{*}{R}\right)}-\left(\eta_3+\overset{*}{T}\right)+k_3\xi_3(t)$$

Linear part of above equations is

$$\dot{\eta}_1=-\eta_1(t)\overset{*}{S}-\frac{\beta}{\gamma}\eta_2(t)\overset{*}{S}+k_1\xi_1(t) \tag{8.1.1}$$

$$\dot{\eta}_2=-\eta_2(t)\overset{*}{R}+\frac{\lambda}{\gamma}\eta_1(t)\overset{*}{R}-\frac{1}{\mu}u_3(t)\overset{*}{R}+k_2\xi_2(t) \tag{8.2.1}$$

$$\dot{\eta}_3=\frac{\rho}{\mu}\eta_2(t)\overset{*}{T}+k_3\xi_3(t) \tag{8.3.1}$$

Taking F.T on both sides of (8.1.1),(8.2.1) and (8.3.1) ,we get

$$K_1\tilde{\xi}_1(\omega)=\left(i\omega+\overset{*}{S}\right)\eta_1(S)+\frac{\beta}{\gamma}\overset{*}{S}\tilde{\eta}_2(\omega) \tag{8.4}$$

$$K_2\tilde{\xi}_2(\omega)=-\frac{\lambda}{\gamma}\overset{*}{R}\tilde{\eta}_1(\omega)+\left(i\omega+\overset{*}{R}\right)\tilde{\eta}_2(\omega)+\frac{1}{\mu}\overset{*}{R}\tilde{\eta}_3(\omega) \tag{8.5}$$

$$K_3\tilde{\xi}_3(\omega)=-\frac{\rho}{\mu}\overset{*}{T}\tilde{\eta}_2(\omega)+i\omega\tilde{\eta}_3(\omega) \tag{8.6}$$

The matrix form of (8.4), (8.5) and (8.6) is $N(\omega)\tilde{\eta}(\omega)=\tilde{\xi}(\omega)$ (8.7)

$$N(\omega)=\begin{bmatrix}a_1(\omega) & b_1(\omega) & c_1(\omega)\\ a_2(\omega) & b_2(\omega) & c_2(\omega)\\ a_3(\omega) & b_3(\omega) & c_3(\omega)\end{bmatrix};\tilde{\eta}(\omega)=\begin{bmatrix}\tilde{\eta}_1(\omega)\\ \tilde{\eta}_2(\omega)\\ \tilde{\eta}_3(\omega)\end{bmatrix};\tilde{\xi}(\omega)=\begin{bmatrix}K_1\tilde{\xi}(\omega)\\ K_2\tilde{\xi}(\omega)\\ K_3\tilde{\xi}(\omega)\end{bmatrix};$$

When

Where

$$a_1(\omega)=i\omega+\overset{*}{S};b_1(\omega)=\frac{\beta}{\gamma}\overset{*}{S};c_1(\omega)=0;a_2(\omega)=\frac{-\lambda}{\gamma}\overset{*}{R};$$
$$b_2(\omega)=\left(i\omega+\overset{*}{R}\right);c_2(\omega)=\frac{1}{\mu}\overset{*}{R};a_3(\omega)=0;b_3(\omega)=-\frac{\rho}{\mu}\overset{*}{T};c_3(\omega)=i\omega. \tag{8.8}$$

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Eq (8.7) can also be written as $\tilde{\eta}(\omega) = \left[N(\omega)\right]^{-1} \tilde{\xi}(\omega)$

Let $\left[N(\omega)\right]^{-1} = L(\omega)$

Therefore, $\tilde{\eta}(\omega) = L(\omega)\tilde{\xi}(\omega)$ (8.8.1)

Where $L(\omega) = \dfrac{Ads(N(\omega))}{|N(\omega)|}$ (8.9)

$|N(\omega)| = R(\omega) + i(\operatorname{Im} g(\omega)).$

From the equation (8.8.1), we have $\tilde{\eta}_i(\omega) = \sum_{j=1}^{3} L_{ij}(\omega)\tilde{\xi}_j(\omega), i = 1,2,3.$ (8.10)

The corresponding spectrum is $S_{\eta_i}(\omega) = \sum_{j=1}^{3} k_j \left|L_{ij}(\omega)\right|^2 ; i = 1,2,3$ (8.11)

The intensities of fluctuations in the variables $\eta_i, i = 1,2,3$ are given by

$$\sigma_{\eta_i}^2 = \frac{1}{2\pi}\sum_{j=1}^{3}\int_{-\infty}^{\infty} k_j \left|L_{ij}(\omega)\right|^2 d\omega; i = 1,2,3. \quad (8.12)$$

By (8.9), we obtain

$$\sigma_{\eta_1}^2 = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} k_1 \left|\frac{A_1}{|N(\omega)|}\right|^2 d\omega + \int_{-\infty}^{\infty} k_2 \left|\frac{B_1}{|N(\omega)|}\right|^2 d\omega + \int_{-\infty}^{\infty} k_3 \left|\frac{C_1}{|N(\omega)|}\right|^2 d\omega\right\} \quad (8.13)$$

$$\sigma_{\eta_2}^2 = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} k_1 \left|\frac{A_2}{|N(\omega)|}\right|^2 d\omega + \int_{-\infty}^{\infty} k_2 \left|\frac{B_2}{|N(\omega)|}\right|^2 d\omega + \int_{-\infty}^{\infty} k_3 \left|\frac{C_2}{|N(\omega)|}\right|^2 d\omega\right\} \quad (8.14)$$

$$\sigma_{\eta_3}^2 = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} k_1 \left|\frac{A_3}{|N(\omega)|}\right|^2 d\omega + \int_{-\infty}^{\infty} k_2 \left|\frac{B_2}{|N(\omega)|}\right|^2 d\omega + \int_{-\infty}^{\infty} k_3 \left|\frac{C_3}{|N(\omega)|}\right|^2 d\omega\right\} \quad (8.15)$$

Where

$$|N(\omega)| == -\left(\omega^2\left(\overset{*}{S} + \overset{*}{R}\right) - \frac{\rho \overset{*}{S}\overset{*}{T}\overset{*}{R}}{\mu^2}\right) - i\left[\omega^3 - \omega \overset{*}{S}\overset{*}{R} + \frac{\omega\rho + \overset{*}{R}}{\mu^2} - \frac{\beta\lambda \overset{*}{S}\overset{*}{R}\overset{*}{\omega}}{\gamma^2}\right].$$

And

$$\left|A_1(\omega)\right|^2 = \left(\frac{\rho \overset{*}{R}\overset{*}{T}}{\mu^2} - \omega^2\right)^2 + \left(\omega \overset{*}{R}\right)^2 ; \left|B_1(\omega)\right|^2 = \left(\frac{\beta\omega \overset{*}{S}}{\gamma}\right)^2 ; \left|C_1(\omega)\right|^2 = \left(\frac{\beta \overset{*}{S}\overset{*}{R}}{\gamma\mu}\right)^2$$

$$\left|A_2(\omega)\right|^2 = \left(\frac{\lambda\omega \overset{*}{R}}{\gamma}\right)^2 ; \left|B_2(\omega)\right|^2 = \omega^2 + \left(\omega \overset{*}{S}\right)^2 ; \left|C_2(\omega)\right|^2 = \left(S/_{\mu}\overset{*}{R}\right)^2 + \left(\omega/_{\mu}\overset{*}{R}\right)^2$$

$$\left|A_3(\omega)\right|^2 = \left(\frac{\lambda \int \overset{*}{T}\overset{*}{R}}{\gamma\mu}\right)^2 ; \left|B_3(\omega)\right|^2 = \left(\frac{\overset{*}{S}\int \overset{*}{T}}{\mu}\right)^2 + \left(\frac{\omega \int \overset{*}{T}}{\mu}\right)^2 ; \left|C_3(\omega)\right|^2 = \left(\overset{*}{R}\overset{*}{S}\left(1 + \frac{\beta\lambda}{\gamma^2}\right) - \omega^2\right)^2 + \left(\omega\left(\overset{*}{S} + \overset{*}{R}\right)\right)^2$$

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$$X_1=\left(\frac{\rho\overset{*}{R}\overset{*}{T}}{\mu^2}-\omega^2\right),Y_1=\omega\overset{*}{R};X_2=0,Y_2=\left(\frac{\beta\omega\overset{*}{S}}{\gamma}\right);X_3=\left(\frac{\beta\overset{*}{S}\overset{*}{R}}{\gamma\mu}\right),Y_3=0;$$

$$X_4=(0),Y_4=\left(\frac{\lambda\omega\overset{*}{R}}{\gamma}\right);X_5=\omega,Y_5=\left(\omega\overset{*}{S}\right);X_6=\left(\frac{S\overset{*}{R}}{\mu}\right),Y_6=\frac{\omega\overset{*}{R}}{\mu};$$

$$X_7=\frac{\lambda S\overset{*}{T}\overset{*}{R}}{\gamma\mu},Y_7=(0);X_8=\omega,Y_8=\left(\frac{\omega S\overset{*}{T}}{\mu}\right);X_9=\left(\frac{S\overset{*}{R}}{\mu}\right),Y_9=\omega\left(\overset{*}{S}+\overset{*}{R}\right).$$

The equations (8.13),(8.14),(8.15) becomes

$$\sigma_{\eta_1}^2=\frac{1}{2\pi}\left\{\int_{-\infty}^{\infty}\frac{1}{R^2(\omega)+I^2(\omega)}\left[K_1\left\{X_1^2+Y^2{}_1\right\}+K_2\left\{X_2^2+Y_2^2\right\}+K_3\left\{X_3^2+Y_3^2\right\}\right]\right\}d\omega$$

$$\sigma_{\eta_2}^2=\frac{1}{2\pi}\left\{\int_{-\infty}^{\infty}\frac{1}{R^2(\omega)+I^2(\omega)}\left[K_1\left\{X_4^2+Y^2{}_4\right\}+K_2\left\{X_5^2+Y_5^2\right\}+K_3\left\{X_6^2+Y_6^2\right\}\right]\right\}d\omega$$

$$\sigma_{\eta_3}^2=\frac{1}{2\pi}\left\{\int_{-\infty}^{\infty}\frac{1}{R^2(\omega)+I^2(\omega)}\left[K_1\left\{X_7^2+Y_7^2\right\}+K_2\left\{X_8^2+Y_8^2\right\}+K_3\left\{X_9^2+Y_9^2\right\}\right]\right\}d\omega$$

If we are interested in the dynamics of system (8.1),(8.2),(8.3) with either $K_1=0$ (or) $K_2=0$ (or) $K_3=0$, Then the population variance are

If $K_1=0$; $K_2=0$

$$\sigma_{\eta_1}^2=\frac{K_3\left(X_3^2\right)}{2\pi}\int_{-\infty}^{\infty}\frac{1}{R^2(\omega)+I^2(\omega)}d\omega$$

$$\sigma_{\eta_2}^2=\frac{K_3\left(X_6^2+Y_6^2\right)}{2\pi}\int_{-\infty}^{\infty}\frac{1}{R^2(\omega)+I^2(\omega)}d\omega$$

$$\sigma_{\eta_3}^2=\frac{K_3\left(X_9^2+Y_9^2\right)}{2\pi}\int_{-\infty}^{\infty}\frac{1}{R^2(\omega)+I^2(\omega)}d\omega$$

If $K_2=0$, $K_3=0$ then

$$\sigma_{\eta_1}^2=\frac{K_1\left(X_1^2+Y_1^2\right)}{2\pi}\int_{-\infty}^{\infty}\frac{1}{R^2(\omega)+I^2(\omega)}d\omega$$

$$\sigma_{\eta_2}^2=\frac{K_1\left(Y_4^2\right)}{2\pi}\int_{-\infty}^{\infty}\frac{1}{R^2(\omega)+I^2(\omega)}d\omega$$

$$\sigma_{\eta_3}^2=\frac{K_1\left(X_7^2\right)}{2\pi}\int_{-\infty}^{\infty}\frac{1}{R^2(\omega)+I^2(\omega)}d\omega$$

If $K_1=0$, $K_3=0$, then

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$$\sigma_{\eta_1}^2 = K_2 \frac{\left(Y_2^{\ 2}\right)}{2\pi} \int_{-\infty}^{\infty} \frac{1}{R^2(\omega)+I^2(\omega)} d\omega$$

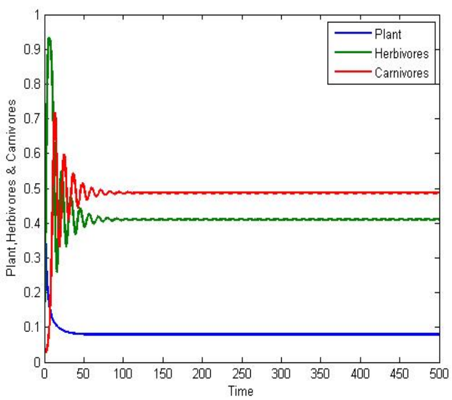
$$\sigma_{\eta_2}^2 = K_2 \frac{\left(X_5^{\ 2}+Y_5^{\ 2}\right)}{2\pi} \int_{-\infty}^{\infty} \frac{1}{R^2(\omega)+I^2(\omega)} d\omega$$

$$\sigma_{\eta_3}^2 = K_2 \frac{X_8^{\ 2}+Y_8^{\ 2}}{2\pi} \int_{-\infty}^{\infty} \frac{1}{R^2(\omega)+I^2(\omega)} d\omega$$

Population variance cannot be easily evaluated though analytical evaluation and hence it has been numerically evaluated for different set of values and parameters using MATLAB. The graphical presentation has been shown in Figurs.9-16.

IX. Numerical Illustrations

In this section, the variation of P.H and C with respect to time and in between P,H,C are computed numerically for a wide range of values of the characterizing parameters $\alpha, \beta, \gamma, \lambda, \delta, \mu, \rho$ as shown in the following using MAT-LAB and the results obtained are illustrated in Figures.



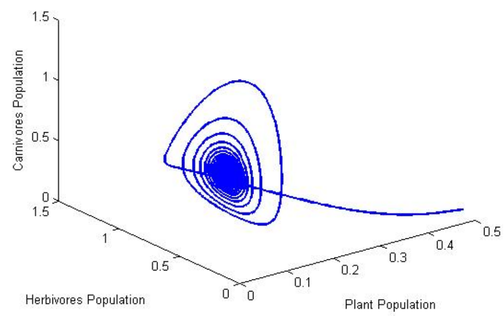
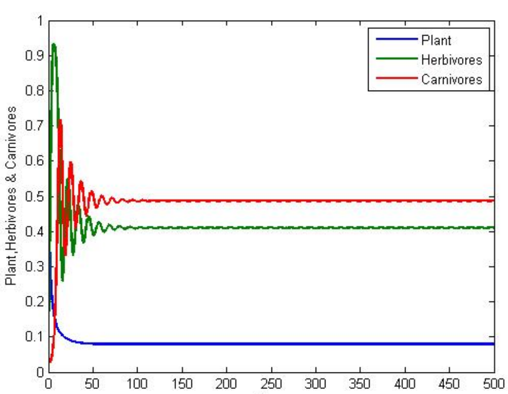


Fig. 1: Steady State E_5 with α =0.08, β = 0.0025, γ =0.985, δ = 1, λ = 2.6123, ρ = 2, μ = 0.41.

Fig. 2: Plant-Herbivores and Carnivores Stable Limit Cycle



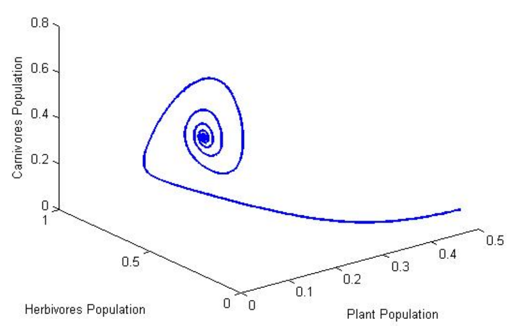


Fig. 3: Steady State E_5 with α =0.0069,

Fig. 4: Plant-Herbivores and Carnivores Stable Limit Cycle

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β = 0.0025, γ =0.985, δ = 1, λ = 2,

ρ = 2, μ = 0.41.

Using Theorem 6.1 we have determined the critical value of λ as λ^* =2.75. The system is found to be unstable for $\lambda > \lambda^*$ around the positive steady state E_5 .when the value of λ is taken to be λ^* =2.75 the solution of the system (2.2) has been as shown in Fig: 6, which explains the instability of system around the positive equilibrium point E_5 .

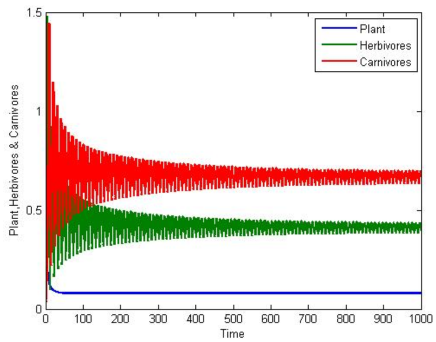

Fig. 5

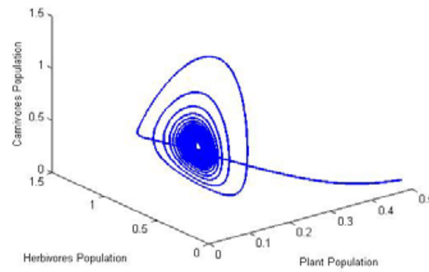

Fig. 6: Bifurcation Diagram for λ = 2.9523

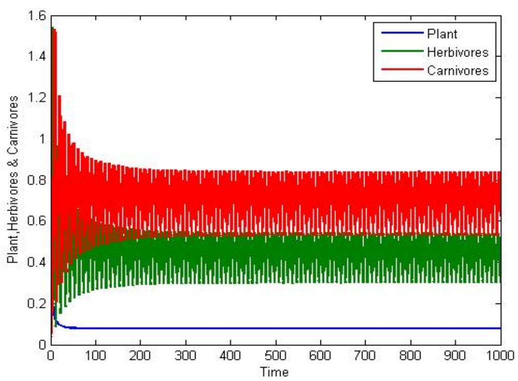

Fig. 7

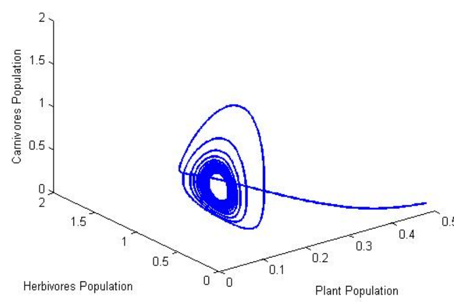

Fig. 8: Bifurcation Diagram for λ = 3.2803

Figures 9-16, demonist rates the periodic behaviour of the growth of the population in a random surroundings with high and low intensities.

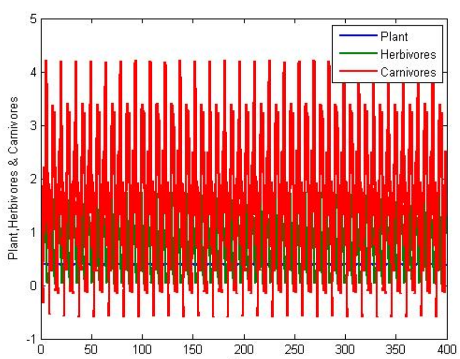

Fig. 9

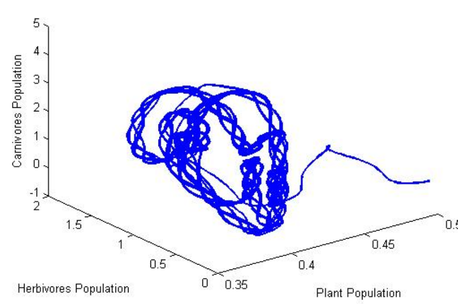

Fig. 10: Variation of P, H and C

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Fig. 9 represents Variation of P,H,C Vs t for α =0.00698, β = 0.0039, γ =0.985, δ = 1, λ = 2, ρ = 2, μ = 0.481 , ξ_1 =0.9332, $\xi_2 = 0.6958, \xi_3 = 0.8940,$ $K_1 = 0.2, K_2 = 0.1, K_3 = 3.$

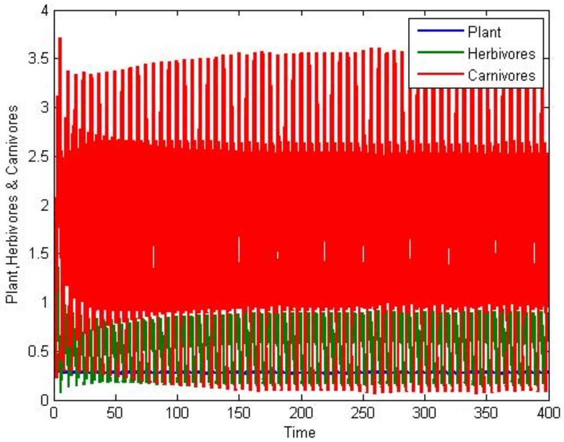

Fig. 11

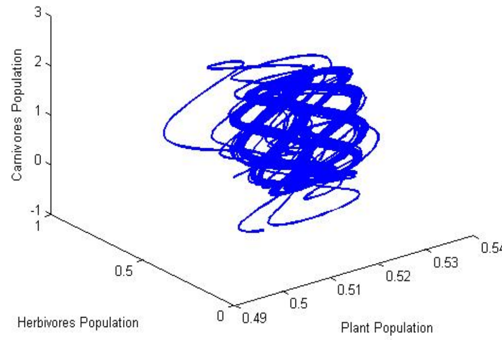

Fig. 12: Variation of P, H and C

Fig.11 represents Variation of P,H,C Vs t for α =0.08, β = 0.0025, γ =1.985, δ = 1, λ = 0.9523, ρ = 2, μ = 0.412 , ξ_1 = 0.952, $\xi_2 = 0.7175, \xi_3 = 0.9511, K_1 = 0.3, K_2 = 0.2, K_3 = 3$

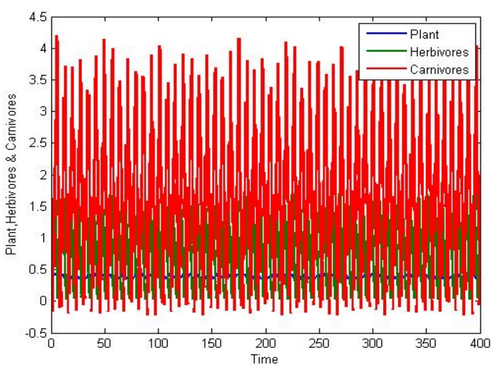

Fig. 13

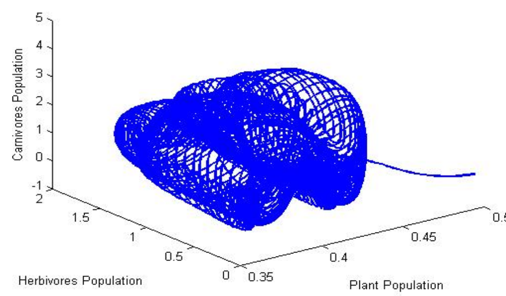

Fig. 14: Variation of P, H and C.

Fig.13 represents Variation of P,H,C Vs t for values α =0.0069, β = 0.0025, γ =0.985, δ = 1, λ = 4, ρ = 2, μ = 0.421, ξ_1 = 0.8837, $\xi_2 = 0.6958, \xi_3 = 0.8012, K_1 = 0.2, K_2 = 0.1, K_3 = 3.$

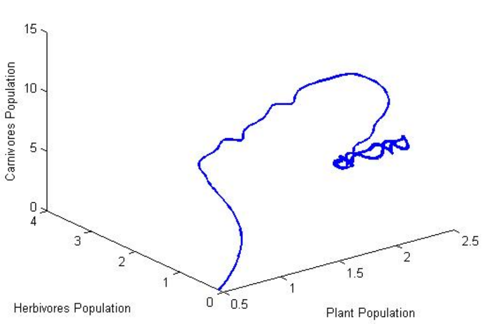

Fig. 15: Variation of P, H and C

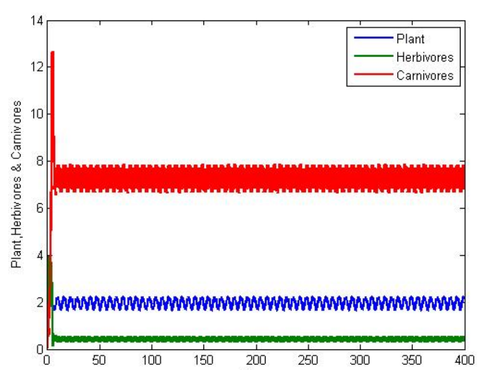

Fig.16

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Fig.16 represents Variation of P,H,C Vs, t for α =0.0069, β = 0.0025, γ =0.985, δ = 1, λ = 4, ρ = 2, μ = 0.421, ξ_1 = 0.8837, $\xi_2 = 0.5978, \xi_3 = 0.930, K_1 = 5, K_2 = 3, K_3 = 2$.

X. Conclusions

In this paper, a plant-herbivore-carnivore ecosystem has been considered. The boundedness of the solutions and existence of steady states is established. The local and global stability of the proposed model around its steady states has been analyzed. A Hopf bifurcation is studied around the positive steady state and the condition for the persistence is established. We have formulated the stochastic version of the model by incorporating the Gaussian White noise under the influence of fluctuating environment. Further we established the behavior of the system with effect of stochastic perturbations. In this stochastic process we observed that the sensitivity of parameters causes large environmental fluctuations which leads to chaotic behavior.

Statement of conflict of interest: On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

[1] Holling.C.S, The functional response of predator to prey density and its role in mimicry and population regulation, Mem. Ent. Can. 45, 1-60 (1965).

[2] Collings. J.: Bifurcation and stability analysis of a temperature dependent mite predator-prey interaction model incorporating peyrefuge, Bulletin of Mathematical Biology, 1995, 57: 63-76.

[3] Debasis. M: Bifurcation and Stability Analysis of a Prey-predator System with a Reserved Area: World Journal of Modelling and Simulation. Vol.8 (2012) No.4, pp. 285-292.

[4] Freedman. H, Wolkowicz. G.: Predator-preysysytems with group defense: The paradox of enrichment revisited. Bulletin of Mathematical Biology, 1986, 48:493-508.

[5] Freedman. H, Waltman. P: Persistence in models of three interacting predator-prey populations, Mathematical Biosciences, 1984, 68: 213-231.

[6] K. Das, N. H. Gazi, “Structural Stability Analysis of an Algal Bloom Mathematical Model in Trophic Interaction,” International Journal of Non-linear Analysis: Real World Applications, Vol. 11, No. 4, 2010, pp. 2191- 2206.

[7] Liu, W.M: Criterion of Hopf bifurcation without using eigenvalues, j. Math. Anal.Appl.182, 250(1994).

[8] Leah Edelstein –keshet: Mathematical Models in Biology, Society for Industrial and Applied Mathematics.

[9] Mukhopadhyay. B., Battacharyya. R.: On a three-tier ecological food chain model with deterministic and random harvesting: A mathematical study, Nonlinear Analysis: Modelling and Control, 2011, Vol.16, No.1, 77-88.

[10] Murray. J.D.: Mathematical Biology II: Spatial Models and Biomedical Applications, Third Edition, Springer.

[11] Panja. P., Mondal. S.K.: Stability analysis of coexistence of three species prey-predator mode. Nonlinear Dyn.

[12] Phani Kumar.N., Pattabhiramacharyulu. N. Ch.: A Three Species Ecosystem Consisting of a Prey, Predator and a Host Commensal to the Prey, Int. J. Open Problems Compt. Math., Vol. 3, No. 1, March 2010.

[13] Ranjith Kumar Upadhyay, Sharada Nandan Raw, Complex dyanamics of a three species food-chain model with Holling type IV functional response; Nonlinear Analysis Modeling and Control,2011,vol.16.No.3.353-374.

[14] Phani Kumar. N.: Global stability of a commensal- host ecological model with limited resources and both are harvesting at a constant rate, Journal of Experimental Sciences 2012, 3(2): 49-52.

[15] Phani Kumar. N., Acharyulu. K.V.L.N.: Global stability of a commensal - host ecological model with limited resources and both are harvesting at a constant rate, International Journal of Mathematics and Computer Applications Research (IJMCAR), Vol. 3, Issue 1, Mar 2013, 95-102.

[16] Rob J.de Boer, Modelling Population Dynamics: a Graphical Approach.

[17] Shuwen. Z., Lansun. C., A. Holling type-II functional response food chain model with impulsive perturbations, chaos, solutions and fractals 24, 1269-1278 (2005).

[18] Strogatz H.S.: Nonlinear Dynamics and Chaos with applications to Physics, Biology, Chemistry, and Engineering, Perseus Books.

[19] Wiggins. S: Introduction to applied nonlinear dyanamical systems and chaos.1990.,

Received: 02 Nov 2018/Accepted: 17 Apr 2019

[20] H. I. Freedman and P. Waltman, "Persistence in models of three interacting predator-prey populations," Mathematical Biosciences, vol. 68, no. 2, pp. 213–231, 1984.

[21] R.M. May,: Stability in randomly fluctuating deterministic environment., Am. Nat. 107 (1973) 621-650.

[22] J. Ripa, P. Lundburg, V. Kaitala,; A general theory of environment noise in ecological food webs,Am,Nat.107(1998)256-263.

[23] Hariyanto, Lukman Hanafil and Suhud Wahyudi1,: Density and Persistence analyze on the spreeding models multi regions multipaches, AIP Conf. Proc. 1867, 020061-1–020061-5.

[24] Xin-You Meng Hai-Feng Huo: Stability and Hopf bifurcation in a three-species system with feedback delays., Nonlinear Dyn (2011) 64:349–364.

[25] Hale, J.K.: Theory of Functional Differential Equations. Springer, New York (1977).

[26] Zhang, J.F., Li, W.T., Yan, X.P.: Multiple bifurcation in a delayed predator–prey diffusion system with a functional response. Nonlinear Anal. RWA 11, 2708–2725 (2010).

Received: 02 Nov 2018/Accepted: 17 Apr 2019