Study of solutions of Commensalism Models by Homotopy-Perturbation Method (HPM)

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Abstract: -In this article, He's Homotopy-perturbation method is used to figure a close estimation to the solution of an ecological model having commensalism collaboration nature between two communicating species, which is given by the system of first order non-linear coupled ordinary differential equations governing in the problem. The numerical results obtained by employing the Homotopy-Perturbation Method (HPM) and the classical fourth order Runge-Kutta (RK) method technique are compared. The HPM method is straight forward, highly effective and a promising tool for the approximate analytical solution of non-linear ODE's. A few plots are introduced to emphasis the reliability of HPM.
Keywords: Homotopy-perturbation method, non-linear differential equations, commensalism, commensal, host, Monod model.

Section: 1

1.1 Introduction

In applied mathematics and science, the non-linear phenomena play a crucial role, and it is one of the most stimulating and particularly active areas of the research. In the research literature of past few decades[1-6] great progress was made in the development of methods for obtaining approximate analytical solutions of non-linear differential equations arising in various fields of Science and Engineering . It is observed that most of these methods require a tedious analysis. Comparatively, He's homotopy perturbation method is better suited to find the approximate solutions of the non-linear differential equations with less effort.

The Homotopy- Perturbation Method (HPM) was initially proposed by Chinese mathematician J.H.He [2-4]. The key thought of this strategy is to reach the actual solution from initial approximate solution as the homotopy parameter, say p, varies from 0 to 1. According to this method the solution is obtained as the summation of an infinite series, which converges to exact solution. Very recently HPM was employed for solving singular second ordered differential equations [7, 8], non-linear population dynamics models [1, 2, 9, 10] and epidemic models [6].The HPM is useful to obtain exact or approximate solutions of linear and non-linear differential equations. No compelling reason to linearization or discretization, computational work and round-off errors is evaded. It has been used to solve effectively, easily and accurately a large class of non-linear problems with approximations. The approximations converge rapidly to exact solutions [11].

The aim of this paper is to explore the He's homotopy-perturbation strategy to three different non-linear biological commensalism models and study their solutions. A comparative study of these solutions with the solutions obtained with 4th order Runge- Kutta method (R-K method) is discussed.

In the three models considered in this paper are related to interaction of commensalism type between the two species. In Model-1& Model-3 the two species have limited food resource and both have logistic growth rate in the absence of the other. The Model-3 is more complex than Model-1 in the sense that the commensalism is characterised by a function of host species say $F\left(N_2\right)$.Unlike Model-1& Model-3 ,in Model-2 the host species follows logistic growth and has limited resources where as the commensal species decline in the absence of host species The survival of this commensal species is only because of the interaction of the host species. The details of these models are given in Section 2.

Section: 2

Model-1: The first model considered is with commensalism between two species utilizing the same limited assets for inborn development of species. Also the interaction of these two species benefits the commensal species. Suppose $N_1 = N_1(t), N_2 = N_2(t)$ represents the size of population of the commensal and host species respectively at any time‘t’; a_1, a_2 are the intrinsic growth rates of the two species; a_{11}, a_{22} are the rates of reduction of the two species due to the limitation of natural resources and a_{12} is commensal coefficient of the interaction of the two species. All these parameters assume non-negative values .The mathematical model governing the commensalism between two species is given by coupled non-linear differential equations,

$$\frac{dN_1}{dt} = a_1N_1 - a_{11}N_1^2 + a_{12}N_1N_2$$

$$\frac{dN_2}{dt} = a_2N_2 - a_{22}N_2^2 \qquad (1)$$

Phani Kumar.etal. [13] Studied the Model-1in which the commensal species (N_1), in spite of the limitation of its natural resources, flourishes by drawing strength from the host species (N_2). It is noticed that the system has four equilibrium states and the co-existent state $\left(\frac{a_1a_{22} + a_2a_{12}}{a_{11}\,a_{22}}, \frac{a_2}{a_{22}}\right)$ is the only stable state under the assumption

(A) $a_1a_{22} + a_2a_{12} \neq a_2a_{22}$, irrespective of the initial values of the two species. The global stability is analyzed by employing a suitably constructed Liapunov’s function.

Model-2: The second model is concerned with the commensalism of two species in which a species (N_1) is weak to sustain, despite of the support of the other host species (N_2) .The host species have its limited food source and the species (N_1) benefits by the interaction with the host species (N_2).Mathematically this model is represented with:

$$\frac{dN_1}{dt} = -d_1N_1 - a_{11}N_1^2 + a_{12}N_1N_2$$

$$\frac{dN_2}{dt} = a_2N_2 - a_{22}N_2^2 \qquad (2)$$

Here $-d_1$ represents the natural death rate of the commensal species in the absence of(N_2). Seshagiri Rao et.al [14] studies the Model-2 extensively and concluded that (i) N_2 will sustain forever in the absence of N_1 and tend to the equilibrium point $\left(0, \frac{a_2}{a_{22}}\right)$ (ii) In the coexistent state the equilibrium point $\left(\frac{a_2a_{12} - d_1a_{11}a_{22}}{a_{11}\,a_{22}}, \frac{a_2}{a_{22}}\right)$ is stable with the condition (B) $d_1 < \frac{a_2a_{12}}{a_{11}a_{22}}$ irrespective of initial conditions. The global stability is analyzed by employing a suitably constructed Liapunov’s function.

Model-3: The third model is generalised model than Model 1 in the sense that the coefficient of commensal is of Monod function $F(N_2)$ [12]. instead of linear function $a_{12}N_2$.This model is represented by

$$\frac{dN_1}{dt} = a_1N_1 - a_{11}N_1^2 + N_1F(N_2)$$

$$\frac{dN_2}{dt} = a_2N_2 - a_{22}N_2^2 \qquad \text{where}$$

$$F(N_2) = \frac{\alpha N_2}{\beta + N_2} \qquad (3)$$

Here the function $F(N_2)$ is the characteristic of the commensal N_1 with respect to the host N_2 with the properties: $F(N_2)$ is bounded and $F(N_2) \to$ a constant $\alpha = F(\infty) > 0$ as $N_2 \to \infty$. Further $\beta(\neq 0)$ is a parameter signifying the strength of the commensalism: $\beta > 0$ strong commensalism, $\beta < 0$ weak commensalism and $\beta = 0$,the interaction would be neutral. Also, here $K_i = \frac{a_i}{a_{ii}}, i = 1,2$ are the carrying capacities of commensal and host species. It is noticed that the system has four equilibrium states and the co-existent state $\left(\bar{N}_1, \bar{N}_2\right) = \left(\frac{1}{a_{11}}\left[K_1a_{11} + \frac{\alpha K_2}{\beta + K_2}\right], K_2\right)$ is always stable under the assumption (C)

$K_1 a_{11} + \frac{\alpha K_2}{\beta + K_2} \neq K_2 a_{22}$ irrespective of the initial values of the two species.

The criterion for asymptotic stability is established by adopting Liapunov technique.

Section: 3

3.1. Homotopy-perturbation method (HPM)

The homotopy-perturbation method is a combination of the classical perturbation technique and homotopy technique. To explain the basic idea of homotopy-perturbation method for solving non-linear differential equations, we consider the following non-linear differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{4}$$

subject to the boundary condition

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma$$

where A is a general differential operator, B a boundary operator, $f(r)$ is a known analytic function, Γ is the boundary of the domain Ω and $\frac{\partial}{\partial n}$ denotes differentiation along the normal drawn outwards from Ω.

In general the operator A, be divided into two parts: a linear part L and a non-linear part N. Therefore equation (4) is written as follows.

$$L(u) - N(u) - f(r) = 0 \tag{5}$$

By the Homotopy technique [4], one constructs a homotopy $v(r,p): \Omega \times [0,1] \to \mathbb{R}$ which satisfies

$$H(v,p) = (1-p)\left[L(v) - L(u_0)\right] + p\left[A(v) - f(r)\right] = 0, \; p \in [0,1], r \in \Omega \tag{6}$$

which is equivalent to

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p\left[N(v) - f(r)\right] = 0 \tag{7}$$

where $p \in [0,1]$ is an imbedding parameter ,and u_0 is an initial approximation of equation(4),which satisfies the boundary conditions. Then equations (6), (7) follow that

$$H(v,0) = L(v) - L(u_0) = 0 \tag{8}$$

and $$H(v,1) = A(v) - f(r) = 0 \tag{9}$$

Thus the changing process of p from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$.

Assume that the solutions of the equations (6) and (7) can be written as a power series in p :

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + p^4 v_4 + ---------- \tag{10}$$

The approximate solution of equation (4) can be obtained as

$$u = \underset{p \to 1}{Lt}\, v = v_0 + v_1 + v_2 + v_3 + v_4 + ---------- \tag{11}$$

3.2. Solutions of the models by HPM

In this section we will apply the HPM to non-linear ordinary differential system (1), (2) and (3).

3.2.1. Solution of Model-1

Consider the system of equations (1) with initial approximations as

$$\begin{aligned} v_{10}(t) &= N_{10}(t) = v_1(0) = c_1 \\ v_{20}(t) &= N_{20}(t) = v_2(0) = c_2 \end{aligned} \tag{12}$$

As explained in section 2, by HPM, we write equations (1) as

$$\begin{aligned} &v_1' - N_{10}' + p\left(N_{10}' - a_1 v_1 + a_{11} v_1^2 - a_{12} v_1 v_2\right) = 0 \\ &v_2' - N_{20}' + p\left(N_{20}' - a_2 v_2 + a_{22} v_2^2\right) = 0 \end{aligned} \tag{13}$$

Assume the approximate solutions of solutions of $N_1(t), N_2(t)$ be

$$\begin{aligned} v_1(t) &= v_{1,0} + pv_{1,1}(t) + p^2 v_{1,2}(t) + p^3 v_{1,3}(t) + p^4 v_{1,4}(t) + ---------- \\ v_2(t) &= v_{2,0} + pv_{2,1}(t) + p^2 v_{2,2}(t) + p^3 v_{2,3}(t) + p^4 v_{2,4}(t) + ---------- \end{aligned} \tag{14}$$

where $v_{i,j}\,(i = 1,2;\; j = 1,2,3,4,........)$ are functions to be determined. Substituting equations (12) and (14) into equation (13) and arranging the terms in the order of increasing powers of p we have

$$\begin{aligned}
&\left[v_{1,1}'(t) + N_{10}'(t) - a_1 v_{1,0}(t) + a_{11} v_{1,0}^2(t) - a_{12} v_{1,0}(t) v_{2,0}(t)\right] p \\
&+\left[v_{1,2}'(t) - a_1 v_{1,1}(t) + 2a_{11} v_{1,0}(t) v_{1,1}(t) - a_{12} v_{1,0}(t) v_{2,1}(t) - a_{12} v_{1,1}(t) v_{2,0}(t)\right] p^2 \\
&+\begin{bmatrix} v_{1,3}'(t) - a_1 v_{1,2}(t) + 2a_{11} v_{1,0}(t) v_{1,2}(t) + a_{11} v_{1,1}^2(t) - a_{12} v_{1,0}(t) v_{2,2}(t) \\ -a_{12} v_{1,1}(t) v_{2,1}(t) - a_{12} v_{1,2}(t) v_{2,0}(t) \end{bmatrix} p^3 \\
&+\begin{bmatrix} v_{1,4}'(t) - a_1 v_{1,3}(t) + 2a_{11} v_{1,0}(t) v_{1,3}(t) + 2a_{11} v_{1,1}(t) v_{1,2}(t) \\ -a_{12} v_{1,0}(t) v_{2,3}(t) - a_{12} v_{1,1}(t) v_{2,2}(t) - a_{12} v_{1,2}(t) v_{2,1}(t) \\ -a_{12} v_{1,3}(t) v_{2,0}(t) \end{bmatrix} p^4 + --\infty = 0 \\
&\left[v_{2,1}'(t) + N_{20}'(t) - a_2 v_{2,0}(t) + a_{22} v_{2,0}^2(t)\right] p \\
&+\left[v_{2,2}'(t) - a_2 v_{2,1}(t) + 2a_{22} v_{2,0}(t) v_{2,1}(t)\right] p^2 \\
&+\left[v_{2,3}'(t) - a_2 v_{2,2}(t) + 2a_{22} v_{2,0}(t) v_{2,2}(t) + a_{22} v_{2,1}^2(t)\right] p^3 \\
&+\left[v_{2,4}'(t) - a_2 v_{2,3}(t) + 2a_{22} v_{2,0}(t) v_{2,3}(t) + 2a_{22} v_{2,1}(t) v_{2,2}(t)\right] p^4 + ------ = 0
\end{aligned} \tag{15}$$

To obtain the unknowns $v_{i,j}(t), i = 1,2;\; j = 1,2,3,4$ we solve the following system of linear differential equations, with the initial conditions given in (12)

From (15)

$$v_{1,1}' + N_{10}' - a_1 v_{1,0} + a_{11} v_{1,0}^2 - a_{12} v_{1,0} v_{2,0} = 0, \quad v_{1,1}(0) = 0 \tag{16}$$

$$v_{2,1}' + N_{20}' - a_2 v_{2,0} + a_{22} v_{2,0}^2 = 0, \quad v_{2,1}(0) = 0 \tag{17}$$

$$v_{1,2}' - a_1 v_{1,1} + 2a_{11} v_{1,0} v_{1,1} - a_{12} v_{1,0} v_{2,1} - a_{12} v_{1,1} v_{2,0} = 0, \quad v_{1,2}(0) = 0 \tag{18}$$

$$v_{2,2}' - a_2 v_{2,1} + 2a_{22} v_{2,0} v_{2,1} = 0, \quad v_{2,2}(0) = 0 \tag{19}$$

$$v_{1,3}' - a_1 v_{1,2} + 2a_{11} v_{1,0} v_{1,2} + a_{11} v_{1,1}^2 - a_{12} v_{1,0} v_{2,2} - a_{12} v_{1,1} v_{2,1} - a_{12} v_{1,2} v_{2,0} = 0, \quad v_{1,3}(0) = 0 \tag{20}$$

$$v_{2,3}' - a_2 v_{2,2} + 2a_{22} v_{2,0} v_{2,2} + a_{22} v_{2,1}^2 = 0, \quad v_{2,3}(0) = 0 \tag{21}$$

$$v_{1,4}' - a_1 v_{1,3} + 2a_{11} v_{1,0} v_{1,3} + 2a_{11} v_{1,1} v_{1,2} - a_{12} v_{1,0} v_{2,3} - a_{12} v_{1,1} v_{2,2} - a_{12} v_{1,2} v_{2,1} - a_{12} v_{1,3} v_{2,0} = 0, \quad v_{1,4}(0) = 0 \tag{22}$$

$$v_{2,4}' - a_2 v_{2,3} + 2a_{22} v_{2,0} v_{2,3} + 2a_{22} v_{2,1} v_{2,2} = 0, \quad v_{2,4}(0) = 0 \tag{23}$$

Solving the differential equations (16)-(23) we get

$$v_{1,1}(t) = a_1 \int_0^t v_{1,0} dt - a_{11} \int_0^t v_{1,0}^2 dt + a_{12} \int_0^t v_{1,0} v_{2,0} dt = (a_1 - a_{11} c_1 + a_{12} c_2) c_1 t \tag{24}$$

$$v_{2,1}(t) = a_2 \int_0^t v_{2,0} dt - a_{22} \int_0^t v_{2,0}^2 dt = (a_2 - a_{22} c_2) c_2 t \tag{25}$$

$$\begin{aligned} v_{1,2}(t) &= a_1 \int_0^t v_{1,1} dt - 2a_{11} \int_0^t v_{1,0} v_{1,1} dt + a_{12} \int_0^t v_{1,0} v_{2,1} dt + a_{12} \int_0^t v_{1,1} v_{2,0} dt \\ &= \begin{bmatrix} (a_1 - a_{11} c_1 + a_{12} c_2) \\ \left(a_1 c_1 - 2a_{11} c_1^2 + a_{12} c_1 c_2\right) \\ + a_{12} (a_2 - a_{22} c_2) c_1 c_2 \end{bmatrix} \frac{t^2}{2} \end{aligned} \tag{26}$$

$$\begin{aligned} v_{2,2}(t) &= a_2 \int_0^t v_{2,1} dt - 2a_{22} \int_0^t v_{2,0} v_{2,1} dt \\ &= (a_2 - a_{22} c_2) c_2 (a_2 - 2a_{22} c_2) \frac{t^2}{2} \end{aligned} \tag{27}$$

$$\begin{aligned} v_{1,3}(t) &= a_1 \int_0^t v_{1,2} dt - 2a_{11} \int_0^t v_{1,0} v_{1,2} dt - a_{11} \int_0^t v_{1,1}^2 dt \\ &\quad + a_{12} \int_0^t v_{1,0} v_{2,2} dt + a_{12} \int_0^t v_{1,1} v_{2,1} dt + a_{12} \int_0^t v_{1,2} v_{2,0} dt \\ &= (a_1 - 2a_{11} c_1 + a_{12} c_2) \begin{bmatrix} (a_1 - a_{11} c_1 + a_{12} c_2) \\ \left(a_1 c_1 - 2a_{11} c_1^2 + a_{12} c_1 c_2\right) \\ + a_{12} (a_2 - a_{22} c_2) c_1 c_2 \end{bmatrix} \frac{t^3}{6} \\ &\quad + (a_1 - a_{11} c_1 + a_{12} c_2) c_1 \begin{bmatrix} a_{12} (a_2 - 2a_{22} c_2) c_2 \\ - a_{11} c_1 (a_1 - a_{11} c_1 + a_{12} c_2) \end{bmatrix} \frac{t^3}{3} \\ &\quad + a_{12} c_1 c_2 (a_2 - a_{22} c_2)(a_2 - 2a_{22} c_2) \frac{t^3}{6} \end{aligned} \tag{28}$$

$$\begin{aligned} v_{2,3}(t) &= a_2 \int_0^t v_{2,2} dt - 2a_{22} \int_0^t v_{2,0} v_{2,2} dt - a_{22} \int_0^t v_{2,1}^2 dt \\ &= (a_2 - a_{22} c_2) c_2 \begin{bmatrix} (a_2 - 2a_{22} c_2)^2 \\ - 2a_{22} c_2 (a_2 - a_{22} c_2) \end{bmatrix} \frac{t^3}{6} \end{aligned} \tag{29}$$

$$\begin{aligned} v_{1,4}(t) &= a_1 \int_0^t v_{1,3} dt - 2a_{11} \int_0^t v_{1,0} v_{1,3} dt - 2a_{11} \int_0^t v_{1,1} v_{1,2} dt \\ &\quad + a_{12} \int_0^t v_{1,0} v_{2,3} dt + a_{12} \int_0^t v_{1,1} v_{2,2} dt \\ &\quad + a_{12} \int_0^t v_{1,2} v_{2,1} dt + a_{12} \int_0^t v_{1,3} v_{2,0} dt \\ &= (a_1 - 2a_{11} c_1 + a_{12} c_2) \left\{ \begin{array}{l} (a_1 - 2a_{11} c_1 + a_{12} c_2) \\ \begin{bmatrix} (a_1 - a_{11} c_1 + a_{12} c_2) \\ \left(a_1 c_1 - 2a_{11} c_1^2 + a_{12} c_1 c_2\right) \\ + a_{12} (a_2 - a_{22} c_2) c_1 c_2 \end{bmatrix} \\ + 2(a_1 - a_{11} c_1 + a_{12} c_2) c_1 \\ \begin{bmatrix} a_{12} (a_2 - a_{22} c_2) c_2 \\ - a_{11} c_1 (a_1 - a_{11} c_1 + a_{12} c_2) \end{bmatrix} \\ + a_{12} c_1 c_2 (a_2 - a_{22} c_2)(a_2 - 2a_{22} c_2) \end{array} \right\} \frac{t^4}{24} \\ &\quad + (a_1 - a_{11} c_1 + a_{12} c_2) c_1 \left\{ \begin{array}{l} a_{12} c_2 (a_2 - a_{22} c_2)(a_2 - 2a_{22} c_2) - \\ 2a_{11} \begin{bmatrix} (a_1 - a_{11} c_1 + a_{12} c_2) \\ \left(a_1 c_1 - 2a_{11} c_1^2 + a_{12} c_1 c_2\right) \\ + a_{12} (a_2 - a_{22} c_2) c_1 c_2 \end{bmatrix} \end{array} \right\} \frac{t^4}{8} \\ &\quad + a_{12} c_1 c_2 (a_2 - a_{22} c_2) \begin{bmatrix} (a_2 - 2a_{22} c_2)^2 \\ - 2a_{22} c_2 (a_2 - a_{22} c_2) \end{bmatrix} \frac{t^4}{24} \\ &\quad + a_{12} c_2 (a_2 - a_{22} c_2) \begin{bmatrix} (a_1 - a_{11} c_1 + a_{12} c_2) \\ \left(a_1 c_1 - 2a_{11} c_1^2 + a_{12} c_1 c_2\right) \\ + a_{12} (a_2 - a_{22} c_2) c_1 c_2 \end{bmatrix} \frac{t^4}{8} \end{aligned} \tag{30}$$

$$\begin{aligned} v_{2,4}(t) &= a_2 \int_0^t v_{2,3} dt - 2a_{22} \int_0^t v_{2,0} v_{2,3} dt - 2a_{22} \int_0^t v_{2,1} v_{2,2} dt \\ &= (a_2 - 2a_{22} c_2)(a_2 - a_{22} c_2) c_2 \\ &\quad \left[(a_2 - 2a_{22} c_2)^2 - 8a_{22} c_2 (a_2 - a_{22} c_2) \right] \frac{t^4}{24} \end{aligned} \tag{31}$$

The approximate solution of $N_1(t), N_2(t)$ is given by

$$N_1(t) = \underset{p \to 1}{Lt}\, v_1(t) = \sum_{k=0}^{4} v_{1,k}(t) \tag{32}$$

$$N_2(t) = \underset{p \to 1}{Lt}\, v_2(t) = \sum_{k=0}^{4} v_{2,k}(t) \tag{33}$$

which yield

$$N_1(t) = c_1 + (a_1 - a_{11}c_1 + a_{12}c_2)c_1 t + \left[\begin{array}{l} (a_1 - a_{11}c_1 + a_{12}c_2) \\ (a_1c_1 - 2a_{11}c_1^2 + a_{12}c_1c_2) \\ +a_{12}(a_2 - a_{22}c_2)c_1c_2 \end{array}\right]\frac{t^2}{2}$$

$$+\left\{\begin{array}{l} (a_1 - 2a_{11}c_1 + a_{12}c_2)\left[\begin{array}{l} (a_1 - a_{11}c_1 + a_{12}c_2) \\ (a_1c_1 - 2a_{11}c_1^2 + a_{12}c_1c_2) \\ +a_{12}(a_2 - a_{22}c_2)c_1c_2 \end{array}\right] \\ \quad +2(a_1 - a_{11}c_1 + a_{12}c_2)c_1 \\ \left[a_{12}(a_2 - 2a_{22}c_2)c_2 - a_{11}c_1(a_1 - a_{11}c_1 + a_{12}c_2)\right] \\ +a_{12}c_1c_2(a_2 - a_{22}c_2)(a_2 - 2a_{22}c_2) \end{array}\right\}\frac{t^3}{6}$$

$$+\left\{\begin{array}{l} (a_1 - 2a_{11}c_1 + a_{12}c_2)\left\{\begin{array}{l} (a_1 - 2a_{11}c_1 + a_{12}c_2) \\ \left[\begin{array}{l} (a_1 - a_{11}c_1 + a_{12}c_2)(a_1c_1 - 2a_{11}c_1^2 + a_{12}c_1c_2) \\ +a_{12}(a_2 - a_{22}c_2)c_1c_2 \end{array}\right] \\ +2(a_1 - a_{11}c_1 + a_{12}c_2)c_1\left[\begin{array}{l} a_{12}(a_2 - a_{22}c_2)c_2 \\ -a_{11}c_1(a_1 - a_{11}c_1 + a_{12}c_2) \end{array}\right] \\ +a_{12}c_1c_2(a_2 - a_{22}c_2)(a_2 - 2a_{22}c_2) \end{array}\right\} \\ \quad +3(a_1 - a_{11}c_1 + a_{12}c_2)c_1\left\{\begin{array}{l} a_{12}c_2(a_2 - a_{22}c_2)(a_2 - 2a_{22}c_2) \\ -2a_{11}\left[\begin{array}{l} (a_1 - a_{11}c_1 + a_{12}c_2) \\ (a_1c_1 - 2a_{11}c_1^2 + a_{12}c_1c_2) \\ +a_{12}(a_2 - a_{22}c_2)c_1c_2 \end{array}\right] \end{array}\right\} \\ \quad +a_{12}c_1c_2(a_2 - a_{22}c_2)\left[(a_2 - 2a_{22}c_2)^2 - 2a_{22}c_2(a_2 - a_{22}c_2)\right] \\ +3a_{12}c_2(a_2 - a_{22}c_2)\left[\begin{array}{l} (a_1 - a_{11}c_1 + a_{12}c_2) \\ (a_1c_1 - 2a_{11}c_1^2 + a_{12}c_1c_2) + a_{12}(a_2 - a_{22}c_2)c_1c_2 \end{array}\right] \end{array}\right\}\frac{t^4}{24} \tag{34}$$

$$N_2(t) = c_2 + (a_2 - a_{22}c_2)c_2 t + (a_2 - a_{22}c_2)c_2(a_2 - 2a_{22}c_2)\frac{t^2}{2}$$
$$+(a_2 - a_{22}c_2)c_2\left[(a_2 - 2a_{22}c_2)^2 - 2a_{22}c_2(a_2 - a_{22}c_2)\right]\frac{t^3}{6}$$
$$+(a_2 - 2a_{22}c_2)(a_2 - a_{22}c_2)c_2$$
$$\left[(a_2 - 2a_{22}c_2)^2 - 8a_{22}c_2(a_2 - a_{22}c_2)\right]\frac{t^4}{24} \tag{35}$$

The numerical simulations of the solutions (34) and (35) with the values of the coefficients satisfied by the condition (A), as `a1=1.15;a11=0.05;a12=0.2;a2=1.78;a22=0.01;N101=2;N20=2` are drawn using HPM and 4th order Runge Kutta method .It is observed from figures1 and 2, that the two methods agree with the solutions for N_1 and N_2 .

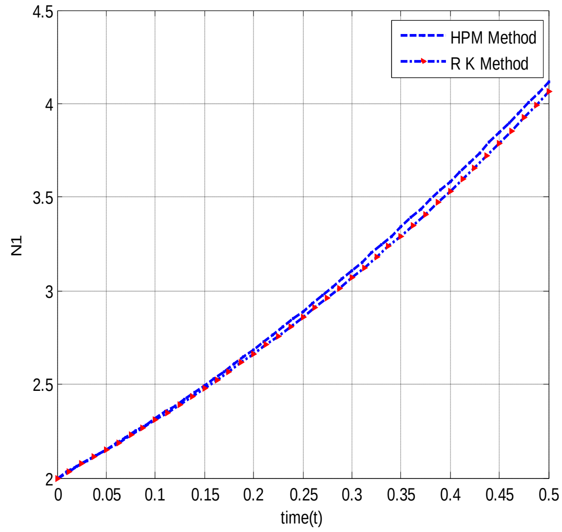

Fig.1. Comparison between the four-term HPM solution of the system in Eq.(1) and the numerical solution with `a1=1.15;a11=0.05;a12=0.2;a2=1.78;a22=0.01;N101=2;N20=2`

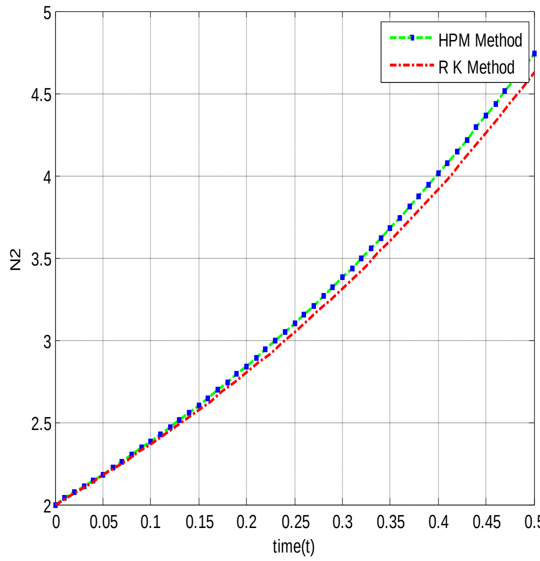

Fig.2. Comparison between the four-term HPM solution of the system in Eq.(1) and the numerical solution with `a1=1.15;a11=0.05;a12=0.2;a2=1.78;a22=0.01;N101=2;N20=2`

3.2. 2 Solution of Model-2

Consider the system of equations (2) with initial approximations as

$$\begin{aligned} v_{10}(t) &= N_{10}(t) = v_1(0) = c_1 \\ v_{20}(t) &= N_{20}(t) = v_2(0) = c_2 \end{aligned} \tag{36}$$

As explained in section 2, by HPM, we write equations (2) as

$$v_1' - N_{10}' + p\left(N_{10}' - d_1 v_1 + a_{11} v_1^2 - a_{12} v_1 v_2\right) = 0$$
$$v_2' - N_{20}' + p\left(N_{20}' - a_2 v_2 + a_{22} v_2^2\right) = 0 \qquad (37)$$

Assume the approximate solutions of solutions of $N_1(t), N_2(t)$ be

$$v_1(t) = v_{1,0} + p v_{1,1}(t) + p^2 v_{1,2}(t) + p^3 v_{1,3}(t) + p^4 v_{1,4}(t) + ----------$$
$$v_2(t) = v_{2,0} + p v_{2,1}(t) + p^2 v_{2,2}(t) + p^3 v_{2,3}(t) + p^4 v_{2,4}(t) + ---------- \qquad (38)$$

where $v_{i,j}\,(i = 1,2;\, j = 1,2,3,4,........)$ are functions to be determined. Substituting equations (36) and (38) into equation (37) and arranging the terms in the order of increasing powers of p we have

$$\left[v_{1,1}'(t) + N_{10}'(t) + d_1 v_{1,0}(t) + a_{11} v_{1,0}^2(t) - a_{12} v_{1,0}(t) v_{2,0}(t)\right] p$$
$$+\begin{bmatrix} v_{1,2}'(t) + d_1 v_{1,1}(t) + 2a_{11} v_{1,0}(t) v_{1,1}(t) \\ -a_{12} v_{1,0}(t) v_{2,1}(t) - a_{12} v_{1,1}(t) v_{2,0}(t) \end{bmatrix} p^2$$
$$+\begin{bmatrix} v_{1,3}'(t) + d_1 v_{1,2}(t) + 2a_{11} v_{1,0}(t) v_{1,2}(t) + a_{11} v_{1,1}^2(t) \\ -a_{12} v_{1,0}(t) v_{2,2}(t) - a_{12} v_{1,1}(t) v_{2,1}(t) - a_{12} v_{1,2}(t) v_{2,0}(t) \end{bmatrix} p^3$$
$$+\begin{bmatrix} v_{1,4}'(t) + d_1 v_{1,3}(t) + 2a_{11} v_{1,0}(t) v_{1,3}(t) \\ +2a_{11} v_{1,1}(t) v_{1,2}(t) - a_{12} v_{1,0}(t) v_{2,3}(t) - a_{12} v_{1,1}(t) v_{2,2}(t) \\ -a_{12} v_{1,2}(t) v_{2,1}(t) - a_{12} v_{1,3}(t) v_{2,0}(t) \end{bmatrix} p^4 + --- = 0$$

$$\left[v_{2,1}'(t) + N_{20}'(t) - a_2 v_{2,0}(t) + a_{22} v_{2,0}^2(t)\right] p$$
$$+\left[v_{2,2}'(t) - a_2 v_{2,1}(t) + 2a_{22} v_{2,0}(t) v_{2,1}(t)\right] p^2$$
$$+\left[v_{2,3}'(t) - a_2 v_{2,2}(t) + 2a_{22} v_{2,0}(t) v_{2,2}(t) + a_{22} v_{2,1}^2(t)\right] p^3 \qquad (39)$$
$$+\begin{bmatrix} v_{2,4}'(t) - a_2 v_{2,3}(t) + 2a_{22} v_{2,0}(t) v_{2,3}(t) \\ +2a_{22} v_{2,1}(t) v_{2,2}(t) \end{bmatrix} p^4 + ------ = 0$$

To obtain the unknowns $v_{i,j}(t), i = 1,2;\, j = 1,2,3,4$ we solve the following system of linear differential equations, with the initial conditions given in (36) From (39)

$$v_{1,1}' + N_{10}' + d_1 v_{1,0} + a_{11} v_{1,0}^2 - a_{12} v_{1,0} v_{2,0} = 0, \quad v_{1,1}(0) = 0 \qquad (40)$$

$$v_{2,1}' + N_{20}' - a_2 v_{2,0} + a_{22} v_{2,0}^2 = 0, \quad v_{2,1}(0) = 0 \qquad (41)$$

$$v_{1,2}' + d_1 v_{1,1} + 2a_{11} v_{1,0} v_{1,1} - a_{12} v_{1,0} v_{2,1} - a_{12} v_{1,1} v_{2,0} = 0, \quad v_{1,2}(0) = 0 \qquad (42)$$

$$v_{2,2}' - a_2 v_{2,1} + 2a_{22} v_{2,0} v_{2,1} = 0, \quad v_{2,2}(0) = 0 \qquad (43)$$

$$v_{1,3}' + d_1 v_{1,2} + 2a_{11} v_{1,0} v_{1,2} + a_{11} v_{1,1}^2 - a_{12} v_{1,0} v_{2,2} - a_{12} v_{1,1} v_{2,1} - a_{12} v_{1,2} v_{2,0} = 0, \quad v_{1,3}(0) = 0 \qquad (44)$$

$$v_{2,3}' - a_2 v_{2,2} + 2a_{22} v_{2,0} v_{2,2} + a_{22} v_{2,1}^2 = 0, \quad v_{2,3}(0) = 0 \qquad (45)$$

$$v_{1,4}' + d_1 v_{1,3} + 2a_{11} v_{1,0} v_{1,3} + 2a_{11} v_{1,1} v_{1,2} - a_{12} v_{1,0} v_{2,3} - a_{12} v_{1,1} v_{2,2} - a_{12} v_{1,2} v_{2,1} - a_{12} v_{1,3} v_{2,0} = 0, \quad v_{1,4}(0) = 0 \qquad (46)$$

$$v_{2,4}' - a_2 v_{2,3} + 2a_{22} v_{2,0} v_{2,3} + 2a_{22} v_{2,1} v_{2,2} = 0, \quad v_{2,4}(0) = 0 \qquad (47)$$

Solving the differential equations (40)-(47) we get

$$v_{1,1}(t) = -d_1 \int_0^t v_{1,0}\,dt - a_{11} \int_0^t v_{1,0}^2\,dt + a_{12} \int_0^t v_{1,0} v_{2,0}\,dt = \left(-d_1 - a_{11} c_1 + a_{12} c_2\right) c_1 t \qquad (48)$$

$$v_{2,1}(t) = a_2 \int_0^t v_{2,0}\,dt - a_{22} \int_0^t v_{2,0}^2\,dt = \left(a_2 - a_{22} c_2\right) c_2 t \qquad (49)$$

$$v_{1,2}(t) = -d_1 \int_0^t v_{1,1}\,dt - 2a_{11} \int_0^t v_{1,0} v_{1,1}\,dt + a_{12} \int_0^t v_{1,0} v_{2,1}\,dt + a_{12} \int_0^t v_{1,1} v_{2,0}\,dt$$
$$= \begin{bmatrix} \left(-d_1 - a_{11} c_1 + a_{12} c_2\right)\left(-d_1 c_1 - 2a_{11} c_1^2 + a_{12} c_1 c_2\right) \\ +a_{12}\left(a_2 - a_{22} c_2\right) c_1 c_2 \end{bmatrix} \frac{t^2}{2} \qquad (50)$$

$$v_{2,2}(t) = a_2 \int_0^t v_{2,1}\,dt - 2a_{22} \int_0^t v_{2,0} v_{2,1}\,dt = \left(a_2 - a_{22} c_2\right) c_2 \left(a_2 - 2a_{22} c_2\right) \frac{t^2}{2} \qquad (51)$$

$$v_{1,3}(t) = -d_1 \int_0^t v_{1,2}\,dt - 2a_{11} \int_0^t v_{1,0} v_{1,2}\,dt - a_{11} \int_0^t v_{1,1}^2\,dt + a_{12} \int_0^t v_{1,0} v_{2,2}\,dt + a_{12} \int_0^t v_{1,1} v_{2,1}\,dt + a_{12} \int_0^t v_{1,2} v_{2,0}\,dt$$
$$= \left(-d_1 - 2a_{11} c_1 + a_{12} c_2\right) \begin{bmatrix} \left(-d_1 - a_{11} c_1 + a_{12} c_2\right) \\ \left(-d_1 c_1 - 2a_{11} c_1^2 + a_{12} c_1 c_2\right) \\ +a_{12}\left(a_2 - a_{22} c_2\right) c_1 c_2 \end{bmatrix} \frac{t^3}{6}$$
$$+ \left(-d_1 - a_{11} c_1 + a_{12} c_2\right) c_1 \begin{bmatrix} a_{12}\left(a_2 - 2a_{22} c_2\right) c_2 \\ -a_{11} c_1 \left(-d_1 - a_{11} c_1 + a_{12} c_2\right) \end{bmatrix} \frac{t^3}{3}$$
$$+ a_{12} c_1 c_2 \left(a_2 - a_{22} c_2\right)\left(a_2 - 2a_{22} c_2\right) \frac{t^3}{6} \qquad (52)$$

$$v_{2,3}(t) = a_2 \int_0^t v_{2,2}\,dt - 2a_{22} \int_0^t v_{2,0} v_{2,2}\,dt - a_{22} \int_0^t v_{2,1}^2\,dt$$
$$= \left(a_2 - a_{22} c_2\right) c_2 \begin{bmatrix} \left(a_2 - 2a_{22} c_2\right)^2 \\ -2a_{22} c_2 \left(a_2 - a_{22} c_2\right) \end{bmatrix} \frac{t^3}{6} \qquad (53)$$

$$
\begin{aligned}
v_{1,4}(t) &= -d_1\int_0^t v_{1,3}dt - 2a_{11}\int_0^t v_{1,0}v_{1,3}dt - 2a_{11}\int_0^t v_{1,1}v_{1,2}dt \\
&+ a_{12}\int_0^t v_{1,0}v_{2,3}dt + a_{12}\int_0^t v_{1,1}v_{2,2}dt \\
&+ a_{12}\int_0^t v_{1,2}v_{2,1}dt + a_{12}\int_0^t v_{1,3}v_{2,0}dt \\
&= \left(-d_1 - 2a_{11}c_1 + a_{12}c_2\right)\left\{\begin{array}{l}
\left(-d_1 - 2a_{11}c_1 + a_{12}c_2\right) \\
\left[\begin{array}{l} \left(-d_1 - a_{11}c_1 + a_{12}c_2\right) \\ \left(\begin{array}{l} -d_1c_1 - 2a_{11}c_1{}^2 \\ +a_{12}c_1c_2 \end{array}\right) \\ +a_{12}\left(a_2 - a_{22}c_2\right)c_1c_2 \end{array}\right] \\
+2\left(-d_1 - a_{11}c_1 + a_{12}c_2\right)c_1 \\
\left[\begin{array}{l} a_{12}\left(a_2 - a_{22}c_2\right)c_2 \\ -a_{11}c_1\left(-d_1 - a_{11}c_1 + a_{12}c_2\right) \end{array}\right] \\
+a_{12}c_1c_2\left(a_2 - a_{22}c_2\right)\left(a_2 - 2a_{22}c_2\right)
\end{array}\right\}\frac{t^4}{24} \\
&+\left(-d_1 - a_{11}c_1 + a_{12}c_2\right)c_1\left\{\begin{array}{l}
a_{12}c_2\left(a_2 - a_{22}c_2\right)\left(a_2 - 2a_{22}c_2\right) \\
-2a_{11}\left[\begin{array}{l} \left(-d_1 - a_{11}c_1 + a_{12}c_2\right) \\ \left(-d_1c_1 - 2a_{11}c_1{}^2 + a_{12}c_1c_2\right) \\ +a_{12}\left(a_2 - a_{22}c_2\right)c_1c_2 \end{array}\right]
\end{array}\right\}\frac{t^4}{8} \\
&+a_{12}c_1c_2\left(a_2 - a_{22}c_2\right)\left[\left(a_2 - 2a_{22}c_2\right)^2 - 2a_{22}c_2\left(a_2 - a_{22}c_2\right)\right]\frac{t^4}{24} \\
&+a_{12}c_2\left(a_2 - a_{22}c_2\right)\left[\begin{array}{l} \left(-d_1 - a_{11}c_1 + a_{12}c_2\right) \\ \left(-d_1c_1 - 2a_{11}c_1{}^2 + a_{12}c_1c_2\right) \\ +a_{12}\left(a_2 - a_{22}c_2\right)c_1c_2 \end{array}\right]\frac{t^4}{8}
\end{aligned}
\tag{54}
$$

$$
\begin{aligned}
v_{2,4}(t) &= a_2\int_0^t v_{2,3}dt - 2a_{22}\int_0^t v_{2,0}v_{2,3}dt - 2a_{22}\int_0^t v_{2,1}v_{2,2}dt \\
&= \left(a_2 - 2a_{22}c_2\right)\left(a_2 - a_{22}c_2\right)c_2 \\
&\left[\left(a_2 - 2a_{22}c_2\right)^2 - 8a_{22}c_2\left(a_2 - a_{22}c_2\right)\right]\frac{t^4}{24}
\end{aligned}
\tag{55}
$$

The approximate solution of $N_1(t), N_2(t)$ is given by

$$
N_1(t) = \underset{p\to 1}{Lt}\, v_1(t) = \sum_{k=0}^{4} v_{1,k}(t) \tag{56}
$$

$$
N_2(t) = \underset{p\to 1}{Lt}\, v_2(t) = \sum_{k=0}^{4} v_{2,k}(t) \tag{57}
$$

which yield

$$
\begin{aligned}
N_1(t) &= c_1 + \left(-d_1 - a_{11}c_1 + a_{12}c_2\right)c_1t + \left[\begin{array}{l} \left(-d_1 - a_{11}c_1 + a_{12}c_2\right) \\ \left(-d_1c_1 - 2a_{11}c_1{}^2 + a_{12}c_1c_2\right) \\ +a_{12}\left(a_2 - a_{22}c_2\right)c_1c_2 \end{array}\right]\frac{t^2}{2} \\
&+\left\{\begin{array}{l}
\left(-d_1 - 2a_{11}c_1 + a_{12}c_2\right)\left[\begin{array}{l} \left(-d_1 - a_{11}c_1 + a_{12}c_2\right) \\ \left(-d_1c_1 - 2a_{11}c_1{}^2 + a_{12}c_1c_2\right) \\ +a_{12}\left(a_2 - a_{22}c_2\right)c_1c_2 \end{array}\right] \\
+2\left(-d_1 - a_{11}c_1 + a_{12}c_2\right)c_1 \\
\left[\begin{array}{l} a_{12}\left(a_2 - 2a_{22}c_2\right)c_2 \\ -a_{11}c_1\left(-d_1 - a_{11}c_1 + a_{12}c_2\right) \end{array}\right] \\
+a_{12}c_1c_2\left(a_2 - a_{22}c_2\right)\left(a_2 - 2a_{22}c_2\right)
\end{array}\right\}\frac{t^3}{6} \\
&+\left\{\begin{array}{l}
\left(-d_1 - 2a_{11}c_1 + a_{12}c_2\right)\left\{\begin{array}{l}
\left(-d_1 - 2a_{11}c_1 + a_{12}c_2\right) \\
\left[\begin{array}{l} \left(-d_1 - a_{11}c_1 + a_{12}c_2\right) \\ \left(-d_1c_1 - 2a_{11}c_1{}^2 + a_{12}c_1c_2\right) \\ +a_{12}\left(a_2 - a_{22}c_2\right)c_1c_2 \end{array}\right] \\
+2\left(-d_1 - a_{11}c_1 + a_{12}c_2\right)c_1 \\
\left[a_{12}\left(a_2 - a_{22}c_2\right)c_2 - a_{11}c_1\left(-d_1 - a_{11}c_1 + a_{12}c_2\right)\right] \\
+a_{12}c_1c_2\left(a_2 - a_{22}c_2\right)\left(a_2 - 2a_{22}c_2\right)
\end{array}\right\} \\
+3\left(-d_1 - a_{11}c_1 + a_{12}c_2\right)c_1\left\{\begin{array}{l}
a_{12}c_2\left(a_2 - a_{22}c_2\right)\left(a_2 - 2a_{22}c_2\right) \\
-2a_{11}\left[\begin{array}{l} \left(-d_1 - a_{11}c_1 + a_{12}c_2\right) \\ \left(-d_1c_1 - 2a_{11}c_1{}^2 + a_{12}c_1c_2\right) \\ +a_{12}\left(a_2 - a_{22}c_2\right)c_1c_2 \end{array}\right]
\end{array}\right\} \\
+a_{12}c_1c_2\left(a_2 - a_{22}c_2\right)\left[\left(a_2 - 2a_{22}c_2\right)^2 - 2a_{22}c_2\left(a_2 - a_{22}c_2\right)\right] \\
+3a_{12}c_2\left(a_2 - a_{22}c_2\right)\left[\begin{array}{l} \left(-d_1 - a_{11}c_1 + a_{12}c_2\right) \\ \left(-d_1c_1 - 2a_{11}c_1{}^2 + a_{12}c_1c_2\right) \\ +a_{12}\left(a_2 - a_{22}c_2\right)c_1c_2 \end{array}\right]
\end{array}\right\}\frac{t^4}{24}
\end{aligned}
\tag{58}
$$

$$
\begin{aligned}
N_2(t) &= c_2 + \left(a_2 - a_{22}c_2\right)c_2t + \left(a_2 - a_{22}c_2\right)c_2\left(a_2 - 2a_{22}c_2\right)\frac{t^2}{2} \\
&+\left(a_2 - a_{22}c_2\right)c_2\left[\left(a_2 - 2a_{22}c_2\right)^2 - 2a_{22}c_2\left(a_2 - a_{22}c_2\right)\right]\frac{t^3}{6} \\
&+\left(a_2 - 2a_{22}c_2\right)\left(a_2 - a_{22}c_2\right)c_2\left[\begin{array}{l} \left(a_2 - 2a_{22}c_2\right)^2 \\ -8a_{22}c_2\left(a_2 - a_{22}c_2\right) \end{array}\right]\frac{t^4}{24}
\end{aligned}
\tag{59}
$$

The numerical simulations of the solutions (58) and (59) with the values of the coefficients satisfied by the condition (B), as `d1=0.4;a11=0.1;a12=0.007;a2=1.3;a22=0.15;N10=1;N20=1;` are drawn using HPM and 4th order Runge Kutta method .It is observed from figures3and 4, that the two methods agree with the solutions for $N_1$and N_2

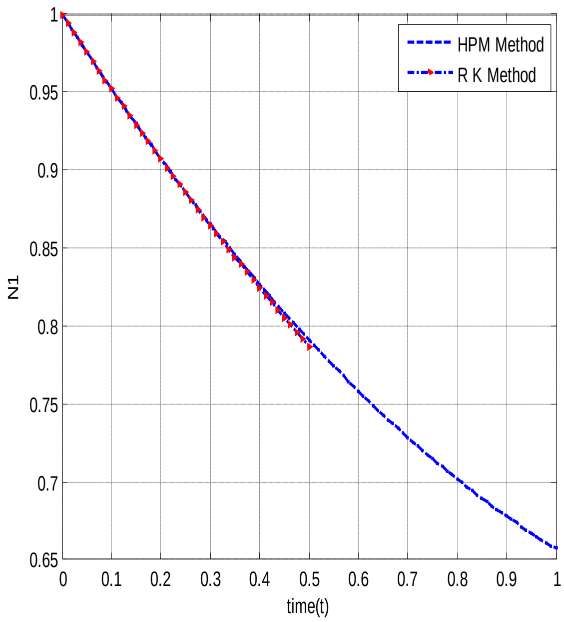

Fig.3. Comparison between the four-term HPM solution of the system in Eq.(2) and the numerical solution with d1=0.4;a11=0.1;a12=0.007;a2=1.3;a22=0.15;N10=1;N20=1

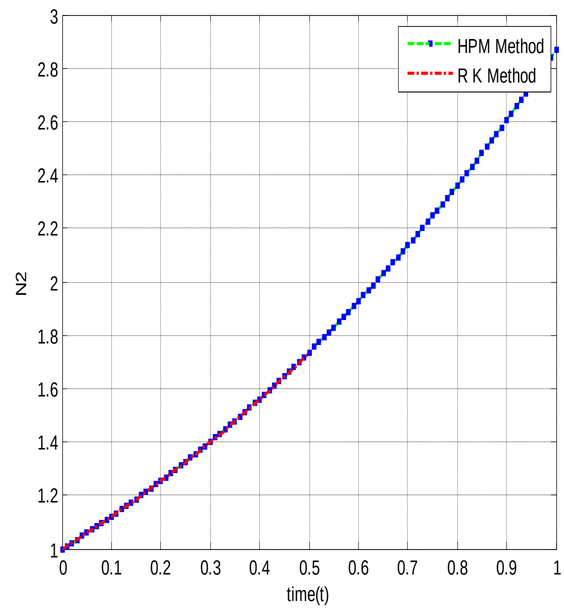

Fig.4. Comparison between the four-term HPM solution of the system in Eq.(2) and the numerical solution with d1=0.4;a11=0.1;a12=0.007;a2=1.3;a22=0.15;N10=1;N20=1

3.2.3 Solution of Model-3

Consider the system of equations (3) with initial approximations as

$$
\begin{aligned}
v_{10}(t) &= N_{10}(t) = v_1(0) = c_1 \\
v_{20}(t) &= N_{20}(t) = v_2(0) = c_2
\end{aligned}
\tag{60}
$$

As explained in section 2, by HPM, we write equations (3) as

$$
\begin{aligned}
&v_1' - N_{10}' + p\left(N_{10}' - a_1 v_1 + a_{11} v_1^2 - \frac{\alpha v_1 v_2}{\beta + v_2}\right) = 0 \\
&v_2' - N_{20}' + p\left(N_{20}' - a_2 v_2 + a_{22} v_2^2\right) = 0
\end{aligned}
\tag{61}
$$

Assume the approximate solutions of solutions of $N_1(t), N_2(t)$ be

$$
\begin{aligned}
v_1(t) &= v_{1,0} + p v_{1,1}(t) + p^2 v_{1,2}(t) + p^3 v_{1,3}(t) + p^4 v_{1,4}(t) + ----------- \\
v_2(t) &= v_{2,0} + p v_{2,1}(t) + p^2 v_{2,2}(t) + p^3 v_{2,3}(t) + p^4 v_{2,4}(t) + -----------
\end{aligned}
\tag{62}
$$

where $v_{i,j}\,(i = 1,2;\ j = 1,2,3,4,........)$ are functions to be determined. Substituting equations (60) and (62) into equation (61) and arranging the terms in the order of increasing powers of p we have

$$
\left\{
\begin{array}{l}
v_{1,1}'(t) + N_{10}'(t) - a_1 v_{1,0}(t) + a_{11} v_{1,0}^2(t) \\
-\dfrac{\alpha}{\beta}\left[\begin{array}{l} v_{1,0}(t) v_{2,0}(t) - \dfrac{1}{\beta} v_{1,0}(t) v_{2,0}^2(t) \\ + \dfrac{1}{\beta^2} v_{1,0}(t) v_{2,0}^3(t) - \dfrac{1}{\beta^3} v_{1,0}(t) v_{2,0}^4(t) \end{array}\right]
\end{array}
\right\} p
$$

$$
+\left\{ v_{1,2}'(t) - a_1 v_{1,1}(t) + 2a_{11} v_{1,0}(t) v_{1,1}(t) - \frac{\alpha}{\beta}
\left[\begin{array}{l}
v_{1,0}(t) v_{2,1}(t) + v_{1,1}(t) v_{2,0}(t) \\
-\dfrac{1}{\beta}\left(\begin{array}{l} 2v_{1,0}(t) v_{2,0}(t) v_{2,1}(t) \\ + v_{1,1}(t) v_{2,0}^2(t) \end{array}\right) \\
+\dfrac{1}{\beta^2}\left(\begin{array}{l} 3v_{1,0}(t) v_{2,1}(t) v_{2,0}^2(t) \\ + v_{1,1}(t) v_{2,0}^3(t) \end{array}\right) \\
-\dfrac{1}{\beta^3}\left(\begin{array}{l} 4v_{1,0}(t) v_{2,0}^3(t) v_{2,1}(t) \\ + v_{1,1}(t) v_{2,0}^4(t) \end{array}\right)
\end{array}\right]
\right\} p^2
$$

$$
+\left\{
\begin{array}{l}
v_{1,3}'(t) - a_1 v_{1,2}(t) + 2a_{11} v_{1,0}(t) v_{1,2}(t) + a_{11} v_{1,1}^2(t) \\
-\dfrac{\alpha}{\beta}\left[\begin{array}{l}
v_{1,0}(t) v_{2,2}(t) + v_{1,1}(t) v_{2,1}(t) + v_{1,2}(t) v_{2,0}(t) - \dfrac{1}{\beta}\left(\begin{array}{l} 2v_{1,0}(t) v_{2,0}(t) v_{2,2}(t) \\ + v_{1,0}(t) v_{2,1}^2(t) \\ + 2v_{1,1}(t) v_{2,0}(t) v_{2,1}(t) \\ + v_{1,2}(t) v_{2,0}^2(t) \end{array}\right) \\
+\dfrac{1}{\beta^2}\left(\begin{array}{l} 3v_{1,0}(t) v_{2,2}(t) v_{2,0}^2(t) + 3v_{1,0}(t) v_{2,0}(t) v_{2,1}^2(t) \\ + 3v_{1,1}(t) v_{2,1}(t) v_{2,0}^2(t) + v_{1,2}(t) v_{2,0}^3(t) \end{array}\right) \\
-\dfrac{1}{\beta^3}\left(\begin{array}{l} 4v_{1,0}(t) v_{2,0}^3(t) v_{2,2}(t) + 6v_{1,0}(t) v_{2,1}^2(t) v_{2,0}^2(t) \\ + 4v_{1,1}(t) v_{2,0}^3(t) v_{2,1}(t) + v_{1,2}(t) v_{2,0}^4(t) \end{array}\right)
\end{array}\right]
\end{array}
\right\} p^3
$$

$$+\left\{\begin{array}{l} v_{1,4}{}'(t)-a_1 v_{1,3}(t)+2a_{11}v_{1,0}(t)v_{1,3}(t) \\ +2a_{11}v_{1,1}(t)v_{1,2}(t)-\dfrac{\alpha}{\beta}\left[\begin{array}{l} v_{1,0}(t)v_{2,3}(t)+v_{1,1}(t)v_{2,2}(t) \\ +v_{1,2}(t)v_{2,1}(t)+v_{1,3}(t)v_{2,0}(t) \\ -\dfrac{1}{\beta}\left(\begin{array}{l} 2v_{1,0}(t)v_{2,0}(t)v_{2,3}(t) \\ +2v_{1,0}(t)v_{2,1}(t)v_{2,2}(t) \\ +2v_{1,1}(t)v_{2,0}(t)v_{2,2}(t) \\ +v_{1,1}(t)v_{2,1}{}^2(t)+2v_{1,2}(t)v_{2,0}(t)v_{2,1}(t) \\ +v_{1,3}(t)v_{2,0}{}^2(t) \end{array}\right) \\ +\dfrac{1}{\beta^2}\left(\begin{array}{l} 3v_{1,0}(t)v_{2,3}(t)v_{2,0}{}^2(t) \\ +6v_{1,0}(t)v_{2,0}(t)v_{2,1}(t)v_{2,2}(t) \\ +v_{1,0}(t)v_{2,1}{}^3(t) \\ +3v_{1,1}(t)v_{2,2}(t)v_{2,0}{}^2(t)+3v_{1,1}(t)v_{2,0}(t)v_{2,1}{}^2(t) \\ +3v_{1,2}(t)v_{2,1}(t)v_{2,0}{}^2(t)+v_{1,3}(t)v_{2,0}{}^3(t) \end{array}\right) \\ -\dfrac{1}{\beta^3}\left(\begin{array}{l} 4v_{1,0}(t)v_{2,0}{}^3(t)v_{2,3}(t)+12v_{1,0}(t)v_{2,1}(t)v_{2,2}(t)v_{2,0}{}^2(t) \\ +4v_{1,0}(t)v_{2,1}{}^3(t)v_{2,0}(t) \\ +4v_{1,1}(t)v_{2,0}{}^3(t)v_{2,2}(t) \\ +6v_{1,1}(t)v_{2,1}{}^2(t)v_{2,0}{}^2(t) \\ +4v_{1,2}(t)v_{2,0}{}^3(t)v_{2,1}(t)+v_{1,3}(t)v_{2,0}{}^4(t) \end{array}\right) \end{array}\right] \end{array}\right\}p^4$$

$$+-----------------------=0$$

$$\begin{aligned} &\left[v_{2,1}{}'(t)+N_{20}{}'(t)-a_2 v_{2,0}(t)+a_{22}v_{2,0}{}^2(t)\right]p \\ &+\left[v_{2,2}{}'(t)-a_2 v_{2,1}(t)+2a_{22}v_{2,0}(t)v_{2,1}(t)\right]p^2 \\ &+\left[v_{2,3}{}'(t)-a_2 v_{2,2}(t)+2a_{22}v_{2,0}(t)v_{2,2}(t)+a_{22}v_{2,1}{}^2(t)\right]p^3 \\ &+\left[\begin{array}{l} v_{2,4}{}'(t)-a_2 v_{2,3}(t)+2a_{22}v_{2,0}(t)v_{2,3}(t) \\ +2a_{22}v_{2,1}(t)v_{2,2}(t) \end{array}\right]p^4+--------=0 \end{aligned} \tag{63}$$

To obtain the unknowns $v_{i,j}(t), i=1,2; j=1,2,3,4$ we solve the following system of linear differential equations, with the initial conditions given in (60)

From (63)

$$v_{1,1}{}'(t)+N_{10}{}'(t)-a_1 v_{1,0}(t)+a_{11}v_{1,0}{}^2(t) -\frac{\alpha}{\beta}\left[\begin{array}{l} v_{1,0}(t)v_{2,0}(t) \\ -\dfrac{1}{\beta}v_{1,0}(t)v_{2,0}{}^2(t) \\ +\dfrac{1}{\beta^2}v_{1,0}(t)v_{2,0}{}^3(t) \\ -\dfrac{1}{\beta^3}v_{1,0}(t)v_{2,0}{}^4(t) \end{array}\right]=0,\ v_{1,1}(0)=0 \tag{64}$$

$$v_{2,1}{}'+N_{20}{}'-a_2 v_{2,0}+a_{22}v_{2,0}{}^2=0,\ \ v_{2,1}(0)=0 \tag{65}$$

$$v_{1,2}{}'(t)-a_1 v_{1,1}(t)+2a_{11}v_{1,0}(t)v_{1,1}(t)-\frac{\alpha}{\beta}\left[\begin{array}{l} v_{1,0}(t)v_{2,1}(t) \\ +v_{1,1}(t)v_{2,0}(t) \\ -\dfrac{1}{\beta}\left(\begin{array}{l} 2v_{1,0}(t)v_{2,0}(t)v_{2,1}(t) \\ +v_{1,1}(t)v_{2,0}{}^2(t) \end{array}\right) \\ +\dfrac{1}{\beta^2}\left(\begin{array}{l} 3v_{1,0}(t)v_{2,1}(t)v_{2,0}{}^2(t) \\ +v_{1,1}(t)v_{2,0}{}^3(t) \end{array}\right) \\ -\dfrac{1}{\beta^3}\left(\begin{array}{l} 4v_{1,0}(t)v_{2,0}{}^3(t)v_{2,1}(t) \\ +v_{1,1}(t)v_{2,0}{}^4(t) \end{array}\right) \end{array}\right]=0, \tag{66}$$

$$v_{1,2}(0)=0$$

$$v_{2,2}{}'-a_2 v_{2,1}+2a_{22}v_{2,0}v_{2,1}=0,\quad v_{2,2}(0)=0 \tag{67}$$

$$v_{1,3}{}'(t)-a_1 v_{1,2}(t)+2a_{11}v_{1,0}(t)v_{1,2}(t)+a_{11}v_{1,1}{}^2(t) -\frac{\alpha}{\beta}\left[\begin{array}{l} v_{1,0}(t)v_{2,2}(t)+v_{1,1}(t)v_{2,1}(t)+v_{1,2}(t)v_{2,0}(t) \\ -\dfrac{1}{\beta}\left(\begin{array}{l} 2v_{1,0}(t)v_{2,0}(t)v_{2,2}(t)+v_{1,0}(t)v_{2,1}{}^2(t) \\ +2v_{1,1}(t)v_{2,0}(t)v_{2,1}(t)+v_{1,2}(t)v_{2,0}{}^2(t) \end{array}\right) \\ +\dfrac{1}{\beta^2}\left(\begin{array}{l} 3v_{1,0}(t)v_{2,2}(t)v_{2,0}{}^2(t)+3v_{1,0}(t)v_{2,0}(t)v_{2,1}{}^2(t) \\ +3v_{1,1}(t)v_{2,1}(t)v_{2,0}{}^2(t)+v_{1,2}(t)v_{2,0}{}^3(t) \end{array}\right) \\ -\dfrac{1}{\beta^3}\left(\begin{array}{l} 4v_{1,0}(t)v_{2,0}{}^3(t)v_{2,2}(t)+6v_{1,0}(t)v_{2,1}{}^2(t)v_{2,0}{}^2(t) \\ +4v_{1,1}(t)v_{2,0}{}^3(t)v_{2,1}(t)+v_{1,2}(t)v_{2,0}{}^4(t) \end{array}\right) \end{array}\right]=0, \tag{68}$$

$$v_{1,3}(0)=0$$

$$v_{2,3}{}'-a_2 v_{2,2}+2a_{22}v_{2,0}v_{2,2}+a_{22}v_{2,1}{}^2=0,\quad v_{2,3}(0)=0 \tag{69}$$

$$v_{1,4}'(t)-a_1v_{1,3}(t)+2a_{11}v_{1,0}(t)v_{1,3}(t)+2a_{11}v_{1,1}(t)v_{1,2}(t)$$
$$-\frac{\alpha}{\beta}\left[\begin{array}{l} v_{1,0}(t)v_{2,3}(t)+v_{1,1}(t)v_{2,2}(t) \\ +v_{1,2}(t)v_{2,1}(t)+v_{1,3}(t)v_{2,0}(t) \\ -\frac{1}{\beta}\begin{pmatrix} 2v_{1,0}(t)v_{2,0}(t)v_{2,3}(t)+ \\ 2v_{1,0}(t)v_{2,1}(t)v_{2,2}(t) \\ +2v_{1,1}(t)v_{2,0}(t)v_{2,2}(t) \\ +v_{1,1}(t)v_{2,1}^2(t) \\ +2v_{1,2}(t)v_{2,0}(t)v_{2,1}(t) \\ +v_{1,3}(t)v_{2,0}^2(t) \end{pmatrix} \\ +\frac{1}{\beta^2}\begin{pmatrix} 3v_{1,0}(t)v_{2,3}(t)v_{2,0}^2(t) \\ +6v_{1,0}(t)v_{2,0}(t)v_{2,1}(t)v_{2,2}(t) \\ +v_{1,0}(t)v_{2,1}^3(t) \\ +3v_{1,1}(t)v_{2,2}(t)v_{2,0}^2(t) \\ +3v_{1,1}(t)v_{2,0}(t)v_{2,1}^2(t) \\ +3v_{1,2}(t)v_{2,1}(t)v_{2,0}^2(t) \\ +v_{1,3}(t)v_{2,0}^3(t) \end{pmatrix} \\ -\frac{1}{\beta^3}\begin{pmatrix} 4v_{1,0}(t)v_{2,0}^3(t)v_{2,3}(t) \\ +12v_{1,0}(t)v_{2,1}(t)v_{2,2}(t)v_{2,0}^2(t) \\ +4v_{1,0}(t)v_{2,1}^3(t)v_{2,0}(t) \\ +4v_{1,1}(t)v_{2,0}^3(t)v_{2,2}(t)+6v_{1,1}(t)v_{2,1}^2(t)v_{2,0}^2(t) \\ +4v_{1,2}(t)v_{2,0}^3(t)v_{2,1}(t)+v_{1,3}(t)v_{2,0}^4(t) \end{pmatrix} \end{array}\right]=0,\quad v_{1,4}(0)=0 \tag{70}$$

$$v_{2,4}'-a_2v_{2,3}+2a_{22}v_{2,0}v_{2,3}+2a_{22}v_{2,1}v_{2,2}=0,\quad v_{2,4}(0)=0 \tag{71}$$

Solving the differential equations (64)-(71) we get

$$v_{1,1}(t)=a_1\int_0^t v_{1,0}dt-a_{11}\int_0^t v_{1,0}{}^2dt+\frac{\alpha}{\beta}\left[\begin{array}{l} \int_0^t v_{1,0}v_{2,0}dt \\ -\frac{1}{\beta}\int_0^t v_{1,0}v_{2,0}{}^2dt \\ +\frac{1}{\beta^2}\int_0^t v_{1,0}v_{2,0}{}^3dt \\ -\frac{1}{\beta^3}\int_0^t v_{1,0}v_{2,0}{}^4(t) \end{array}\right] \tag{72}$$
$$=c_1\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)t$$

$$v_{2,1}(t)=a_2\int_0^t v_{2,0}dt-a_{22}\int_0^t v_{2,0}{}^2dt=(a_2-a_{22}c_2)c_2t \tag{73}$$

$$v_{1,2}(t)=a_1\int_0^t v_{1,1}dt-2a_{11}\int_0^t v_{1,0}v_{1,1}dt+\frac{\alpha}{\beta}\left[\begin{array}{l} \int_0^t v_{1,0}v_{2,1}dt+\int_0^t v_{1,1}v_{2,0}dt \\ -\frac{1}{\beta}\begin{pmatrix} 2\int_0^t v_{1,0}v_{2,0}v_{2,1}dt \\ +\int_0^t v_{1,1}v_{2,0}{}^2dt \end{pmatrix} \\ +\frac{1}{\beta^2}\begin{pmatrix} 3\int_0^t v_{1,0}v_{2,1}v_{2,0}{}^2dt \\ +\int_0^t v_{1,1}v_{2,0}{}^3dt \end{pmatrix} \\ -\frac{1}{\beta^3}\begin{pmatrix} 4\int_0^t v_{1,0}v_{2,0}{}^3v_{2,1}dt \\ +\int_0^t v_{1,1}v_{2,0}{}^4dt \end{pmatrix} \end{array}\right]$$
$$=\left[\begin{array}{l} c_1\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\left(a_1-2a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right) \\ +\frac{\alpha}{\beta}c_1c_2V_2(a_2-a_{22}c_2) \end{array}\right]\frac{t^2}{2} \tag{74}$$

$$v_{2,2}(t)=a_2\int_0^t v_{2,1}dt-2a_{22}\int_0^t v_{2,0}v_{2,1}dt$$
$$=(a_2-a_{22}c_2)c_2(a_2-2a_{22}c_2)\frac{t^2}{2} \tag{75}$$

$$v_{1,3}(t)=a_1\int_0^t v_{1,2}dt-2\int_0^t a_{11}v_{1,0}v_{1,2}dt-\int_0^t a_{11}{v_{1,1}}^2dt$$

$$+\frac{\alpha}{\beta}\left[\begin{array}{l}\int_0^t v_{1,0}v_{2,2}dt+\int_0^t v_{1,1}v_{2,1}dt\\ +\int_0^t v_{1,2}v_{2,0}dt\\ -\frac{1}{\beta}\left(\begin{array}{l}2\int_0^t v_{1,0}v_{2,0}v_{2,2}dt+\int_0^t v_{1,0}{v_{2,1}}^2dt\\ +2\int_0^t v_{1,1}v_{2,0}v_{2,1}dt+\int_0^t v_{1,2}{v_{2,0}}^2dt\end{array}\right)\\ +\frac{1}{\beta^2}\left(\begin{array}{l}3\int_0^t v_{1,0}v_{2,2}{v_{2,0}}^2dt+3\int_0^t v_{1,0}v_{2,0}{v_{2,1}}^2dt\\ +3\int_0^t v_{1,1}v_{2,1}{v_{2,0}}^2dt+\int_0^t v_{1,2}{v_{2,0}}^3dt\end{array}\right)\\ -\frac{1}{\beta^3}\left(\begin{array}{l}4\int_0^t v_{1,0}{v_{2,0}}^3v_{2,2}dt+6\int_0^t v_{1,0}{v_{2,1}}^2{v_{2,0}}^2dt\\ +4\int_0^t v_{1,1}{v_{2,0}}^3v_{2,1}dt+\int_0^t v_{1,2}{v_{2,0}}^4dt\end{array}\right)\end{array}\right]$$

$$=\left\{\begin{array}{l}\left(a_1-2a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\\ \left[\begin{array}{l}c_1\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\\ \left(a_1-2a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\\ +\frac{\alpha}{\beta}c_1c_2V_2\left(a_2-a_{22}c_2\right)\end{array}\right]\end{array}\right\}\frac{t^3}{6} \tag{76}$$

$$-a_{11}\left[{c_1}^2\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)^2\right]\frac{t^3}{3}$$

$$+\left[\frac{\alpha}{\beta}c_1c_2V_2\left(a_2-a_{22}c_2\right)\left(a_2-2a_{22}c_2\right)\right]\frac{t^3}{6}$$

$$+\left[\frac{\alpha}{\beta}c_1c_2V_2\left(a_2-a_{22}c_2\right)\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\right]\frac{t^3}{3}$$

$$+\left[\frac{\alpha}{\beta^2}c_1{c_2}^2V_3\left(a_2-a_{22}c_2\right)^2\right]\frac{t^3}{3}$$

$$v_{2,3}(t)=a_2\int_0^t v_{2,2}dt-2a_{22}\int_0^t v_{2,0}v_{2,2}dt-a_{22}\int_0^t {v_{2,1}}^2dt$$

$$=\left(a_2-a_{22}c_2\right)c_2\left[\begin{array}{l}\left(a_2-2a_{22}c_2\right)^2\\ -2a_{22}c_2\left(a_2-a_{22}c_2\right)\end{array}\right]\frac{t^3}{6} \tag{77}$$

$$v_{1,4}(t)=\int_0^t a_1v_{1,3}dt-2\int_0^t a_{11}v_{1,0}v_{1,3}dt-2\int_0^t a_{11}v_{1,1}v_{1,2}dt$$

$$+\frac{\alpha}{\beta}\left[\begin{array}{l}\int_0^t v_{1,0}v_{2,3}dt+v\int_0^t {}_{1,1}v_{2,2}dt+\int_0^t v_{1,2}v_{2,1}dt+\int_0^t v_{1,3}v_{2,0}dt\\ -\frac{1}{\beta}\left(\begin{array}{l}2\int_0^t v_{1,0}v_{2,0}v_{2,3}dt+2\int_0^t v_{1,0}v_{2,1}v_{2,2}dt\\ +2\int_0^t v_{1,1}v_{2,0}v_{2,2}dt\\ +\int_0^t v_{1,1}{v_{2,1}}^2dt+2\int_0^t v_{1,2}v_{2,0}v_{2,1}dt\\ +\int_0^t v_{1,3}{v_{2,0}}^2dt\end{array}\right)\\ +\frac{1}{\beta^2}\left(\begin{array}{l}3\int_0^t v_{1,0}v_{2,3}{v_{2,0}}^2dt\\ +6\int_0^t v_{1,0}v_{2,0}v_{2,1}v_{2,2}dt+\int_0^t v_{1,0}{v_{2,1}}^3dt\\ +3\int_0^t v_{1,1}v_{2,2}{v_{2,0}}^2dt\\ +3\int_0^t v_{1,1}v_{2,0}{v_{2,1}}^2dt\\ +3\int_0^t v_{1,2}v_{2,1}{v_{2,0}}^2dt+\int_0^t v_{1,3}{v_{2,0}}^3dt\end{array}\right)\\ -\frac{1}{\beta^3}\left(\begin{array}{l}4\int_0^t v_{1,0}{v_{2,0}}^3v_{2,3}dt\\ +12\int_0^t v_{1,0}v_{2,1}v_{2,2}{v_{2,0}}^2dt\\ +4\int_0^t v_{1,0}{v_{2,1}}^3v_{2,0}dt\\ +4\int_0^t v_{1,1}{v_{2,0}}^3v_{2,2}dt+6\int_0^t v_{1,1}{v_{2,1}}^2{v_{2,0}}^2dt\\ +4\int_0^t v_{1,2}{v_{2,0}}^3v_{2,1}dt+\int_0^t v_{1,3}{v_{2,0}}^4dt\end{array}\right)\end{array}\right]$$

$$=\left(a_1-2a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\left\{\begin{array}{l}\left(a_1-2a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\\ \left[\begin{array}{l}c_1\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\left(a_1-2a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)+\\ \frac{\alpha}{\beta}c_1c_2V_2\left(a_2-a_{22}c_2\right)\end{array}\right]\\ -2a_{11}c_1^2\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)^2\\ +\frac{\alpha}{\beta}c_1c_2V_2\left(a_2-a_{22}c_2\right)\left(a_2-2a_{22}c_2\right)\\ +2\frac{\alpha}{\beta}c_1V_2\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\left(a_2-a_{22}c_2\right)c_2\\ +2\frac{\alpha}{\beta^2}c_1c_2^2V_3\left(a_2-a_{22}c_2\right)^2\end{array}\right\}\frac{t^4}{24}$$

$$-2a_{11}c_1\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\left[\begin{array}{l}c_1\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\\ \left(a_1-2a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\\ +\frac{\alpha}{\beta}c_1c_2V_2\left(a_2-a_{22}c_2\right)\end{array}\right]\frac{t^4}{8}$$

$$+\frac{\alpha}{\beta}c_1c_2V_2\left(a_2-a_{22}c_2\right)\left[\left(a_2-2a_{22}c_2\right)^2-2a_{22}c_2\left(a_2-a_{22}c_2\right)\right]$$

$$\frac{t^4}{24}+\frac{\alpha}{\beta}c_1V_2\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)c_2\left(a_2-a_{22}c_2\right)\left(a_2-2a_{22}c_2\right)\frac{t^4}{8}$$

$$+\frac{\alpha}{\beta}V_2\left[\begin{array}{l}c_1\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\\ \left(a_1-2a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)+\frac{\alpha}{\beta}c_1c_2V_2\left(a_2-a_{22}c_2\right)\end{array}\right]c_2\left(a_2-a_{22}c_2\right)\frac{t^4}{8}$$

$$+\frac{\alpha}{\beta^2}2c_1V_3\left(a_2-a_{22}c_2\right)^2c_2^2\left(a_2-2a_{22}c_2\right)\frac{t^4}{8}$$

$$+\frac{\alpha}{\beta^2}c_1V_3\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)$$

$$\left(a_2-a_{22}c_2\right)^2c_2^2\frac{t^4}{4}$$

$$+\frac{\alpha}{\beta^3}c_1\left(1-\frac{4}{\beta}c_2\right)c_2^3\left(a_2-a_{22}c_2\right)^3\frac{t^4}{4} \tag{78}$$

$$v_{2,4}(t)=a_2\int_0^t v_{2,3}dt-2a_{22}\int_0^t v_{2,0}v_{2,3}dt$$

$$-2a_{22}\int_0^t v_{2,1}v_{2,2}dt=\left(a_2-2a_{22}c_2\right)\left(a_2-a_{22}c_2\right)c_2 \tag{79}$$

$$\left[\left(a_2-2a_{22}c_2\right)^2-8a_{22}c_2\left(a_2-a_{22}c_2\right)\right]\frac{t^4}{24}$$

where $V_1=1-\frac{1}{\beta}c_2+\frac{1}{\beta^2}c_2^2-\frac{1}{\beta^3}c_2^3$;

$V_2=1-\frac{1}{\beta}2c_2+\frac{1}{\beta^2}3c_2^2-\frac{1}{\beta^3}4c_2^3$;

$V_3=-1+\frac{1}{\beta}3c_2-\frac{1}{\beta^2}6c_2^2$

If 4-term approximations are sufficient we obtain

$$N_1(t)=\underset{p\to 1}{Lt}\,v_1(t)=\sum_{k=0}^{4}v_{1,k}(t) \tag{80}$$

$$N_2(t)=\underset{p\to 1}{Lt}\,v_2(t)=\sum_{k=0}^{4}v_{2,k}(t) \tag{81}$$

which yield

$$N_1(t)=c_1+c_1\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)t+\left[\begin{array}{l}c_1\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\\ \left(a_1-2a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\\ +\frac{\alpha}{\beta}c_1c_2V_2\left(a_2-a_{22}c_2\right)\end{array}\right]\frac{t^2}{2}$$

$$+\left\{\begin{array}{l}\left(a_1-2a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\left[\begin{array}{l}c_1\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\\ \left(a_1-2a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\\ +\frac{\alpha}{\beta}c_1c_2V_2\left(a_2-a_{22}c_2\right)\end{array}\right]\\ -2a_{11}c_1^2\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)^2\\ +\frac{\alpha}{\beta}c_1c_2V_2\left(a_2-a_{22}c_2\right)\left(a_2-2a_{22}c_2\right)\\ +2\frac{\alpha}{\beta}c_1c_2V_2\left(a_2-a_{22}c_2\right)\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\\ +2\frac{\alpha}{\beta^2}c_1c_2^2V_3\left(a_2-a_{22}c_2\right)^2\end{array}\right\}\frac{t^3}{6}$$

$$+\left(a_1-2a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\left\{\begin{array}{l}\left(a_1-2a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\left[\begin{array}{l}c_1\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\\ \left(a_1-2a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\\ +\frac{\alpha}{\beta}c_1c_2V_2\left(a_2-a_{22}c_2\right)\end{array}\right]\\ -2a_{11}c_1^2\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)^2\\ +\frac{\alpha}{\beta}c_1c_2V_2\left(a_2-a_{22}c_2\right)\left(a_2-2a_{22}c_2\right)\\ +2\frac{\alpha}{\beta}c_1V_2\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\\ \left(a_2-a_{22}c_2\right)c_2+2\frac{\alpha}{\beta^2}c_1c_2^2V_3\left(a_2-a_{22}c_2\right)^2\\ -6a_{11}c_1\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\\ \left[\begin{array}{l}c_1\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\\ \left(a_1-2a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\\ +\frac{\alpha}{\beta}c_1c_2V_2\left(a_2-a_{22}c_2\right)\end{array}\right]\\ +\frac{\alpha}{\beta}c_1c_2V_2\left(a_2-a_{22}c_2\right)\left[\left(a_2-2a_{22}c_2\right)^2-2a_{22}c_2\left(a_2-a_{22}c_2\right)\right]\\ +3\frac{\alpha}{\beta}c_1V_2\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)c_2\left(a_2-a_{22}c_2\right)\left(a_2-2a_{22}c_2\right)\\ +3\frac{\alpha}{\beta}V_2\left[\begin{array}{l}c_1\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\\ \left(a_1-2a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)+\frac{\alpha}{\beta}c_1c_2V_2\left(a_2-a_{22}c_2\right)\end{array}\right]\\ c_2\left(a_2-a_{22}c_2\right)+6\frac{\alpha}{\beta^2}c_1V_3\left(a_2-a_{22}c_2\right)^2c_2^2\left(a_2-2a_{22}c_2\right)\\ 6\frac{\alpha}{\beta^2}c_1V_3\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\\ \left(a_2-a_{22}c_2\right)^2c_2^2+6\frac{\alpha}{\beta^3}c_1\left(1-\frac{4}{\beta}c_2\right)\\ c_2^3\left(a_2-a_{22}c_2\right)^3\end{array}\right\}\frac{t^4}{24} \tag{82}$$

$$N_2(t) = c_2 + (a_2 - a_{22}c_2)c_2t + (a_2 - a_{22}c_2)$$
$$c_2(a_2 - 2a_{22}c_2)\frac{t^2}{2}$$
$$+(a_2 - a_{22}c_2)c_2\begin{bmatrix}(a_2 - 2a_{22}c_2)^2 \\ -2a_{22}c_2(a_2 - a_{22}c_2)\end{bmatrix}\frac{t^3}{6} \quad (83)$$
$$+(a_2 - 2a_{22}c_2)(a_2 - a_{22}c_2)c_2$$
$$\left[(a_2 - 2a_{22}c_2)^2 - 8a_{22}c_2(a_2 - a_{22}c_2)\right]\frac{t^4}{24}$$

The numerical simulations of the solutions (58) and (59) with the values of the coefficients satisfied by the condition (C) as `a1=1;a11=2.4;a2=0.03;a22=0.65;p=2;q=3.1;N101=1;N20=1;` drawn using HPM and 4th order Runge Kutta method .It is observed from figures 5 and 6, that the two methods agree with the solutions for $N_1$and N_2

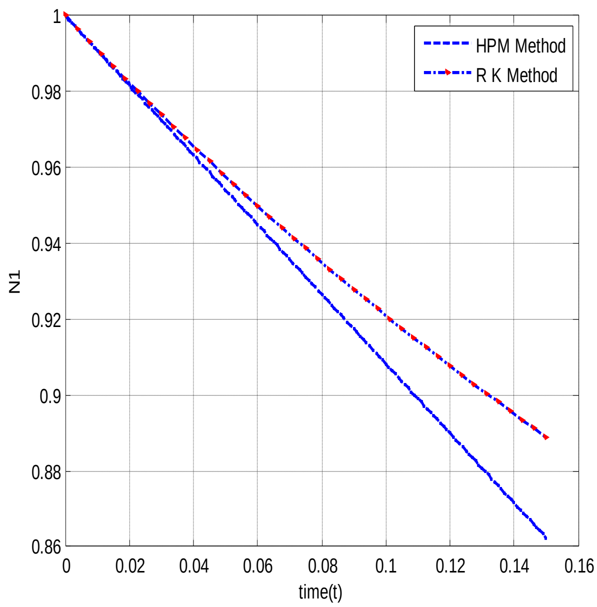

Fig.5. Comparison between the four-term HPM solution of the system in Eq.(3) and the numerical solution with `a1=1;a11=2.4;a2=0.03;a22=0.65;` α `=2;` β `=3.1;N101=1;N20=1`

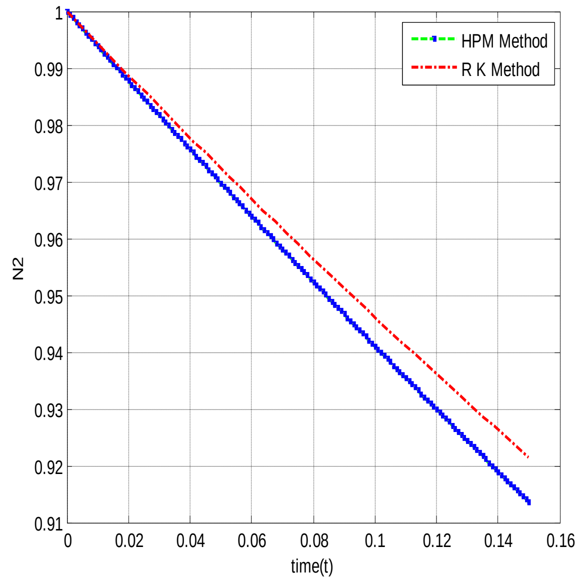

Fig.5. Comparison between the four-term HPM solution of the system in Eq.(3) and the numerical solution with `a1=1;a11=2.4;a2=0.03;a22=0.65;` α `=2;` β `=3.1;N101=1;N20=1`

Section: 4

4.1 Conclusions and results.

In this paper the approximate analytical solutions of continuous population models of interacting species, specifically with commensalism interaction by utilizing He's homotopy perturbation method are evaluated.

Figs.1and 2 show the comparison between the four-term HPM solutions of the system in Eq. (1) and the numerical solutions with `a1=1.15;a11=0.05;a12=0.2;a2=1.78;a22=0.01;N101=2;N20=2`.These numerical solutions are obtained by using ode45, an ordinary differential equation solver found in the *Matlab package* . From both of the figures it is clear that there is a very close approximation between the solutions for N1 (Commensal population) and N2 (Host population) in the time interval [0, 0.45] by using only 4 terms of the series given by Eq. (11), which indicates that the speed of convergence of HPM is very fast. A better approximate analytical solution for $t \geq 0.45$ can be achieved by adding more terms to the series in Eq. (11).

Figs.3 and 4 shows the comparison between the four-term HPM solutions of the system in Eq. (2) and the numerical solutions with `d1=0.4;a11=0.1;a12=0.007;a2=1.3;a22=0.15;N10=1;N20=1`.It seems that the solutions for N1 and N2 almost identical in the time interval [0, 0.5], after $t \geq 0.5$ the speed of convergence by HPM is very fast. A better approximation to the solution is acquired by adding new terms to the series in Eq. (11).

Figs.5 and 6 demonstrates the comparison between the four-term HPM solutions of the system in Eq. (3) and the numerical solutions with a1=1;a11=2.4;a2=0.03;a22=0.65; α=2; β=3.1;N101=1;N20=1.It is again clear that the two figures for the solutions (N_1, N_2)look indistinguishable in the time interval [0.0.38], after $t \geq 0.38$ the speed of convergence by HPM is very fast. A better approximation to the solution is obtained by adding new terms to the series in Eq. (11).

In this paper one clearly conclude that the solutions obtained by the standard Homotopy-Perturbation Method (HPM) were not valid for large time span unless more terms are ascertained.

Acknowledgements

The authors thank University Grants Commission (India) for the financial support in the form of Major Research project (Grant no 42-18/2013(SR)).

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