Special Situation Where the Outer Measure via Order Preserving Valuation on a Relative Sub Lattice Obeys the Outer Measure Generated by Measure

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Abstract. This paper is motivated by GABOR SZASZ's introduction of outer measure based on valuation for lattices. We introduce order-preserving valuation on a relative sub lattice of the lattice, a countable cover of an element in a relative sub lattice, outer measure induced by a valuation of a relative sub lattice, Ɲ* measurability and to establish certain elementary properties of induced outer measure via order preserving valuation on a relative sub lattice. To prove outer measure via order preserving valuation on a relative sub lattice is also valid the outer measure generated by measure, we define the definition of outer measure on relative sub lattices of a lattice L induced by a measure Ɲ on Q of relative sub lattices of L, measure on algebra of relative sub lattices and outer measure of relative sub lattices, we prove $L(\mathscr{B})$ is a sigma-algebra where $L(\mathscr{B})$ is the class of Ɲ* measurable relative sub lattices and deduce a corollary that Ɲ*(S) = Ɲ(S), also we prove that outer measure to any subset E of a relative sub lattice $S \in Q_\sigma$ and outer measure to relative sub lattice $B \in Q_{\sigma\delta}$ are equal. Finally, by defining sigma-finite measure on Q and Ɲ* generated by Ɲ we prove a relative sub lattice E is measurable Ɲ* ⇔ E is the proper difference S ~ B of a relative sub lattice A in $Q_{\sigma\delta}$ and a relative sub lattice B with Ɲ*(B) = 0. Further each relative sub lattice B with Ɲ*(B) = 0 is contained in a relative sub lattice C in $Q_{\sigma\delta}$ with Ɲ*(C) = 0.

INTRODUCTION

By the concepts of Anil Kumar etal [1, 2] here we provideIn section2, the definitions of valuation on lattice, order preserving valuation on a relative sub lattice of lattice, countable cover of an element in a relative sub lattice, outer measure induced by a valuation of a relative sub lattice, Ɲ* measurability are defined and some properties of the induced outer measure are proved. The following results are established.

[1] On a distributive lattice and a valuation of the zero element is zero then it is finitely-additive. For any element of a relative sub lattice the order preserving valuation of an element is equal to its outer measure. Outer measure of zero element is zero. Ɲ* measurability of an element in a C.B.L. Every element of relative sub lattice is Ɲ* measurable. An element Ꮎ in L is measurable ⇔ Ꮎ′ is measurable. If the Ɲ* measurability of an element is zero also, the (set of all) Ɲ* measurable elements of a relative sub lattice H is sigma-algebra

After defining valuation on a relative sub lattice H of a lattice L and the outer measure on L induced by this valuation. To establish outer measure via order preserving valuation on a relative sub lattice is also valid the outer measure generated by measure we define outer measure on relative sub lattices of a lattice L induced by a measure Ɲ on Q (where Q is an algebra) of relative sub lattices of L. One might wonder if this is mere repetition of the contents of the above. However, in this particular situation we get more information than what we have presented in above.

In section3 measure on algebra of relative sub lattices and outer measure of relative sub lattices are defined and also some basic properties of measure are presented.

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In section4 we prove $L(\mathscr{B})$ is a sigma-algebra where $L(\mathscr{B})$ is the class of Ɲ* measurable relative sub lattices and deduce a corollary that Ɲ*(S) = Ɲ(S).

In section5 we prove that the outer measure of any subset E of a relative sub lattice $S \in Q_\sigma$ and the outer measure of relative sub lattice $B \in Q_{\sigma\delta}$ are equal. Finally, by defining sigma-finite measure on Q and Ɲ* generated by Ɲ we prove a relative sub lattice E is Ɲ* measurable ⇔ E is the proper difference S ~ B of a relative sublattice S in $Q_{\sigma\delta}$ and a relative sub lattice B with Ɲ*(B) = 0. Further each relative sub lattice B with Ɲ*(B) = 0 is contained in a relative sub lattice C in $Q_{\sigma\delta}$ with Ɲ*(C) = 0. It is note that partial lattice is called as relative sub lattice.

ORDER PRESERVING VALUATION ON A RELATIVE SUB LATTICE

This section is motivated by GABOR SZASZ'^s**[3]** introduction of outer measure based on valuation for lattices. We introduce the outer measure via order preserving valuation on a relative sub lattice H.

Definition2.1.[1]On L, И: L →□^∞ (Where □^∞ = □∪ {∞}) is called valuation if $И(Ѳ_1 \vee Ѳ_2) + И(Ѳ_1 \wedge Ѳ_2) = И(Ѳ_1) + И(Ѳ_2)$.

И is order preserving if $И(Ѳ_1) \leq И(Ѳ_2)$ whenever $Ѳ_1 \leq Ѳ_2$.

Note2.1. $И(Ѳ_1 \vee Ѳ_2) \leq И(Ѳ_1) + И(Ѳ_2)$.

Example2.1. Any chain (real valued function) is a valuation.

We now define order preserving valuation on a relative sub lattice of a lattice.

Definition2.2. Suppose L is a lattice and H is a relative sub lattice in L. И: L →□^∞ is said to be valuation on H if $И(Ѳ_1 \vee Ѳ_2) + И(Ѳ_1 \wedge Ѳ_2) = И(Ѳ_1) + И(Ѳ_2)$ whenever $Ѳ_1 \wedge Ѳ_2$, $Ѳ_1 \vee Ѳ_2$, $Ѳ_1$, $Ѳ_2$, belongs to H and $И(Ѳ_1) \leq И(Ѳ_2)$ whenever $Ѳ_1 \leq Ѳ_2$.

Example2.2. [Giri Thesis]

Definition2.3. Let H a relative sub lattice in L, (where L be C.B.L) and $\{Ѳ_i \ / \ i \in N\}$ in H is a countable cover of Ѳ in L if $Ѳ \leq \bigvee_{i=1}^{\infty} Ѳ_i$.

Example2.3. L = **P**(N), H = **P**(E) where E is positive even integers. Clearly H is a relative sub lattice of L.

Put Ѳ = E ∈ L, $Ѳ_n = \{2n\}$. Here $E = \vee Ѳ_n$ is countable cover.

Definition2.4. Let И be an order preserving valuation on a relative sub lattice H contained in a C.B.L L.

If Ѳ∈L define $Ɲ^*(Ѳ) = \inf \{\sum И(Ѳ_i) / \{Ѳ_i\}$ countable cover of Ѳ in H }

Clearly И*(Ѳ) = +∞ if Ѳ does not have countable cover in L. Ɲ* is called outer measure induced by H.

Definition2.5. (Ɲ* measurable) An element Ѳ∈L is called Ɲ*measurable if $Ɲ^*(x \wedge Ѳ) + Ɲ^*(x \wedge Ѳ') = Ɲ^*(x)$ for any x∈H.

Observation2.1. [4] If $Ƨ_1$ and $Ƨ_2$ are Ɲ* measurable then

$Ɲ^*(Ƨ_1 \vee Ƨ_2) + Ɲ^*(Ƨ_1 \wedge Ƨ_2) = Ɲ^*(Ƨ_1) + Ɲ^*(Ƨ_2)$.

Result2.1. The following are properties of outer measure induced by H

(i) Ɲ*(Ѳ) is non- negative for every Ѳ ∈H and also Ɲ*(0) = 0

(ii) Ѳ≤ b in L ⇒ Ɲ*(Ѳ) ≤ Ɲ*(b)

(iii) If $\{Ѳ_n\}$ is any sequence in L, $Ɲ^*(\vee Ѳ_n) \leq \sum И^*(Ѳ_n)$

Proof of (iii). A sequence $\{Ѳ_{nƧ}\}$ in H ∋ $Ѳ_n \leq \bigvee_k Ѳ_{nk}$

Then $\{Ѳ_{nk} / n \geq Ѳ, k \geq Ѳ\}$ is a countable collection in H ∋ $\bigvee_n Ѳ_n \leq \bigvee_k Ѳ_{nk}$

$\Rightarrow Ɲ^*(\bigvee_n Ѳ_n) \leq \sum_{n,k} И(Ѳ_{nk})$ -------------- (1)

Now ∀ k, there exist $\{b_{nk}\}$ ∋

$Ѳ_n < \bigvee_k b_{nk}$ and $\sum_k И(b_{nk}) < Ɲ^*(Ѳ_n) + \frac{\varepsilon}{2^n}$

Therefore $\sum_n \sum_k И(b_{nk}) < Ɲ^*(Ѳ_n) + \varepsilon$

By (1), $Ɲ^*(\vee Ѳ_n) \leq \sum_n \sum_k И(b_{nk}) < Ɲ^*(Ѳ_n) + \varepsilon$

The above is true for every ε>0

$\Rightarrow Ɲ^*(\vee Ѳ_n) \leq \sum И^*(Ѳ_n)$.

Result2.2. If $И(0) = 0$ and L is distributive then И is finitely-additive.
Proof. Clearly if $ɵ_1 \wedge ɵ_2 = 0$ then
$И(ɵ_1 \vee ɵ_2) = И(ɵ_1 \vee ɵ_2) + 0$, by definition of 2.2 $И(ɵ_1) + И(ɵ_2) = И(ɵ_1 \wedge ɵ_2) + И(ɵ_1 \vee ɵ_2)$

Assume that if $ɵ_1, ɵ_2, \ldots, ɵ_n$ are pairwise disjoint, $\sum_{i=1}^{n} И(ɵ_i) = И(\bigvee_{i=1}^{n} ɵ_i)$

If $ɵ_1, ɵ_2, \ldots, ɵ_{n+1}$ are pairwise disjoint, $И(\bigvee_{i=1}^{n} ɵ_i) = \sum_{i=1}^{n} И(ɵ_i)$

Now $И(\bigvee_{i=1}^{n+1} ɵ_i) = И(\bigvee_{i=1}^{n} ɵ_i \vee ɵ_{n+1})$

$= И(ɵ_{n+1}) + И(\bigvee_{i=1}^{n} ɵ_i)$

[Since $ɵ_{n+1} \wedge (\bigvee_{i=1}^{n} ɵ_i) = \bigvee_{i=1}^{n+1} (ɵ_{n+1} \wedge ɵ_i) = 0$]

$= \sum_{i=1}^{n} И(ɵ_{n+1}) + И(ɵ_i) = \sum_{i=1}^{n+1} И(ɵ_i)$

Hence by induction И is finitely-additive.

Result2.3. If $ɵ \in H$ then $Ɲ^*(ɵ) = И(ɵ)$

Proof. By definition2.4, $Ɲ^*(ɵ) \leq И(ɵ)$ ------ (1)
Let $l \in H$.
Clearly $И(ɵ) \leq И(\bigvee ɵ_j) \leq \sum И(ɵ_j)$, whenever $ɵ \leq \bigvee ɵ_j$
$\Rightarrow И(ɵ) \leq \inf \sum И(ɵ_j) = Ɲ^*(ɵ)$
$\Rightarrow И(ɵ) \leq Ɲ^*(ɵ)$ ------------------- (2)
(1) and (2) gives $И(ɵ) = Ɲ^*(ɵ)$.
Result2.4. $Ɲ^*(0) = 0$
Proof. By definition 2.4, $И(0) = 0$
By the result 2.3, $Ɲ^*(0) = 0$
Result2.5. An element $ɵ \in L$ is measurable $\Leftrightarrow$
$Ɲ^*(x) \geq Ɲ^*(x \wedge ɵ) + Ɲ^*(x \wedge ɵ') \ \forall\, x \in H$.
Proof. For any $x \in H$, $ɵ \in L$
Now, $x = x \wedge ɵ$
$\Rightarrow x = x \wedge (ɵ \vee ɵ')$
$\Rightarrow x = (x \wedge ɵ) \vee (x \wedge ɵ')$
$\Rightarrow Ɲ^*(x) = Ɲ^*[(x \wedge ɵ) \vee (x \wedge ɵ')]$
$\Rightarrow Ɲ^*(x) \leq Ɲ^*(x \wedge ɵ) + Ɲ^*(x \wedge ɵ')$
[By countable sub-additivity]
So, to prove ɵ is measurable, we prove
$Ɲ^*(x) \geq Ɲ^*(x \wedge ɵ) + Ɲ^*(x \wedge ɵ') \ \forall\, x \in H$.
Result2.6. Every element ɵ of H is Ɲ* measurable and $Ɲ^*(ɵ) = И(ɵ)$.
Proof. Take a Ɲ* measurable element ɵ in H.
By result 2.3, it holds good.

Result2.7. If $ɵ \in L$ then ɵ is measurable $\Leftrightarrow$ ɵ' is measurable
Proof. Suppose ɵ is measurable. If $x \in H$
$\Rightarrow Ɲ^*(x) = Ɲ^*[x \wedge ɵ] + Ɲ^*[x \wedge ɵ']$
$\Rightarrow Ɲ^*(x) = Ɲ^*[x \wedge (ɵ')'] + Ɲ^*[x \wedge ɵ']$
$\Rightarrow Ɲ^*(x) = Ɲ^*[x \wedge ɵ'] + Ɲ^*[x \wedge (ɵ')']$
$\Rightarrow ɵ'$ is measurable.
Result2.8. For any a relative sub lattice H in L, $ɵ \in L \ni Ɲ^*(ɵ) = 0$ then ɵ is measurable.

Proof. Take $ɵ \in L \ni Ɲ^*(ɵ) = 0$.

For any $x \in H$, clearly $x \wedge \ell \leq \ell$

$\Rightarrow N^*(x \wedge \ell) \leq N^*(\ell)$

$\Rightarrow N^*(x \wedge \ell) \leq 0$ ----------------- (1)

But $N^*(x \wedge \ell) \geq 0$ ---------------------- (2)

[Since $N^*(l) \geq 0$]

From (1) and (2),

$N^*(x \wedge \ell) = 0$ --------------------------- (3)

Again, for any ℓ' in H, $x \wedge \ell' \leq x$

$\Rightarrow N^*(x \wedge \ell') \leq N^*(x)$

$\Rightarrow N^*(x) \geq N^*(x \wedge l')$

$\Rightarrow N^*(x) \geq 0 + N^*(x \wedge l')$

$\Rightarrow N^*(x) \geq N^*(x \wedge l) + N^*(x \wedge l')$ [By (3)]

Therefore l is measurable.

Result2.9. If ℓ_1, ℓ_2in L are N^* measurable then so is $\ell_1 \vee \ell_2$.

Proof. Let $\ell_1, \ell_2 \in$ L and $x \in H$.

Consider $N^*[x \wedge (\ell_1 \vee \ell_2) + N^*[x \wedge (\ell_1 \vee \ell_2)']$

$= N^*\{x \wedge [\ell_1 \vee (\ell_2 \wedge \ell_1')]\} + N^*[x \wedge (\ell_1 \vee \ell_2)']$

By distributive property and D'Morgans's law, the proof follows.

Result2.10. If H is a relative sub lattice in Land ℓ_1, ℓ_2in L are N^* measurable then so is $\ell_1 \wedge \ell_2$.

Proof. Let $\ell 1, \ell 2$ are N^* measurable

$\Rightarrow \ell_1', \ell_2'$ are N^* measurable

$\Rightarrow \ell_1' \vee \ell_2' = (\ell_1 \wedge \ell_2)'$ is N^* measurable

$\Rightarrow \ell_1 \wedge \ell_2$ is N^* measurable.

Result2.11. Let x be any element in H and $\ell_1, \ell_2, \ldots\ldots, \ell_n$ be a finite sequence of disjoint N^* measurable elements then

$$N^*[x \wedge (\bigvee_{i=1}^{\infty} \ell_i)] = \sum_{i=1}^{n} N^*(x \wedge \ell_i).$$

Proof. By induction on n.

If n = 1, obviously the above result is true.

Assume that this the above is true for n-1 element.

That is $N^*[x \wedge (\bigvee_{i=1}^{n-1} \ell_i)] = \sum_{i=1}^{n-1} N^*(x \wedge \ell_i)$

We have to show that

$$N^*[x \wedge (\bigvee_{i=1}^{n} \ell_i)] = \sum_{i=1}^{n} N^*(x \wedge \ell_i).$$

Consider

$$N^*[x \wedge (\bigvee_{i=1}^{n} \ell_i)]$$

$$= N^*[\{x \wedge (\bigvee_{i=1}^{n} \ell_i)\} \wedge \ell_n] + N^*[\{x \wedge (\bigvee_{i=1}^{n} \ell_i)\} \wedge \ell_n']$$

[Since ℓ_n is measurable]

$$= N^*[\{x \wedge (\bigvee_{i=1}^{n} \ell_i)\} \wedge \ell_n] + N^*[\{x \wedge (\bigvee_{i=1}^{n} \ell_i)\} \wedge \ell_n']$$

$= \mathcal{N}^*(x \wedge e_n) + \mathcal{N}^*[x \wedge \{(\bigvee_{i=1}^{n-1} e_i) \vee e_n\} \wedge e_n']$

$[\text{Since } e_n \subseteq \bigvee_{i=1}^{n} e_i]$

$= \mathcal{N}^*(x \wedge e_n) + \mathcal{N}^*[x \wedge (\bigvee_{i=1}^{n-1} e_i) \wedge e_n']$

$[\text{Since } e_n \wedge e_n' = \phi]$

$= \mathcal{N}^*(x \wedge e_n) + \mathcal{N}^*[x \wedge (\bigvee_{i=1}^{n-1} e_i)]$

$[\text{Since } e_i \wedge e_n = \phi,\ \forall i = 1,2,\ldots\ldots,n-1$

$\Rightarrow e_i \subseteq e_i',\ \forall i = 1,2,\ldots\ldots n-1$

$\Rightarrow \bigvee_{i=1}^{n-1} e_i \subseteq e_n'\ ]$

$= \mathcal{N}^*(x \wedge e_n) + \sum_{i=1}^{n-1} \mathcal{N}^*(x \wedge e_i)$

$= \sum_{i=1}^{n} \mathcal{N}^*(x \wedge e_i)$, is true for n.

Therefore $\mathcal{N}^*[x \wedge (\bigvee_{i=1}^{n} e_i)] = \sum_{i=1}^{n} \mathcal{N}^*(x \wedge e_i)$ is true $\forall n$ by induction.

Hence the result holds $\forall n$.

Result2.12. The set of all $\mathcal{N}^*$ measurable elements of a relative sub lattice H is sigma-algebra.

Proof. Let M the set of all $\mathcal{N}^*$ measurable elements of a relative sub lattice H.

Let $e \in M$.

$\Rightarrow$, l is $\mathcal{N}^*$ measurable element

By result2.7 e' is $\mathcal{N}^*$ measurable

$\Rightarrow e' \in M$

Now let $e_1, e_2 \in M$

$\Rightarrow$, e_1, e_2 are $\mathcal{N}^*$ measurable elements

$\Rightarrow$, $e_1 \vee e_2$ is $\mathcal{N}^*$ measurable

[By result2.9]

$\Rightarrow$, $e_1 \vee e_2 \in M$

Therefore, M is algebra.

To prove M is sigma-algebra, it is enough to show that countable collection of $\mathcal{N}^*$ measurable elements is also in M

Let $e_1, e_2, \ldots\ldots$ be a collection of $\mathcal{N}^*$ measurable elements, put $e = \bigvee_{n=1}^{\infty} e_n$

Write $b_1 = e_1$ and $b_n = e_n - (\bigvee_{i=1}^{n-1} e_i)$ for $n > 1$

Then $\{e_n\}$ is a class of mutually disjoint $\mathcal{N}^*$ measurable elements $\ni b_n \wedge b_m = 0$ for $n \neq m$ and also $\bigvee_{n=1}^{\infty} b_n = \bigvee_{n=1}^{\infty} e_n = e$.

Write $c_n = \bigvee_{i=1}^{\infty} b_i \forall n$

$\Rightarrow c_n$ is $\mathcal{N}^*$ measurable for every n

To show that e is $\mathcal{N}^*$ measurable, Let x be any element in H

Consider, $\mathcal{N}^*(x) = \mathcal{N}^*(x \wedge c_n) + \mathcal{N}^*(x \wedge c_n')$

[Since c_n is $\mathcal{N}^*$measurable]

$\mathcal{N}^*(x) \geq \mathcal{N}^*(x \wedge c_n) + \mathcal{N}^*(x \wedge e')$ [Since $c_n \leq e. \Rightarrow e' \leq c_n', \Rightarrow x \wedge e' \leq x \wedge c_n'$,

$\Rightarrow \text{Ɲ}^*(x \wedge \text{Ꝺ}') \le \text{Ɲ}^*(x \wedge c_n')]$

$= \text{Ɲ}^*(x \wedge (\bigvee_{i=1}^{n} b_i)) + \text{Ɲ}^*(x \wedge \text{Ꝺ}')$

$= \sum_{i=1}^{n} \text{Ɲ}^*(x \wedge b_i) + \text{Ɲ}^*(x \wedge \text{Ꝺ}')$

[By result 2.11]

$\text{Ɲ}^*(x) \ge \sum_{i=1}^{n} \text{Ɲ}^*(x \wedge b_i) + \text{Ɲ}^*(x \wedge \text{Ꝺ}') \ \forall\, n$

$\text{Ɲ}^*(x) \ge \sum_{i=1}^{\infty} \text{Ɲ}^*(x \wedge b_i) + \text{Ɲ}^*(x \wedge \text{Ꝺ}')$

$\ge \text{Ɲ}^*[\bigvee_{i=1}^{\infty} (x \wedge b_i)] + \text{Ɲ}^*(x \wedge \text{Ꝺ}')$

[Since Ɲ* is countable sub-additive]

$= \text{Ɲ}^*[x \wedge (\bigvee_{i=1}^{\infty} b_i)] + \text{Ɲ}^*(x \wedge \text{Ꝺ}')$

$= \text{Ɲ}^*[x \wedge \text{Ꝺ}] + \text{Ɲ}^*(x \wedge \text{Ꝺ}')$

$\text{Ɲ}^*(x) \ge \text{Ɲ}^*[x \wedge \text{Ꝺ}] + \text{Ɲ}^*(x \wedge \text{Ꝺ}')$

⇒, Ꝺ is Ɲ* measurable

Therefore, countable collections of Ɲ* measurable elements are in M.

Hence M is sigma-algebra.

OUTER MEASURE GENERATED BY MEASURE

In section2, we defined valuation on a relative sub lattice H of a lattice L and the outer measure on L induced by this valuation. To establish outer measure via order preserving valuation on a relative sub lattice is also valid the outer measure generated by measure. Here we provide the definition of outer measure on relative sub lattices of a lattice L induced by a measure Ɲ on Q (where Q is an algebra) of relative sub lattices of L. One might wonder if this is mere repetition of the contents of the section2. However, in this particular situation we get more information than what we have presented in section2.

Let us recall the definition of outer measure.

Definition3.1. [4] Let X be any set, $L \subset \mathbf{P}(X)$ be a lattice $\exists\ \hat{s} \in L \Rightarrow \hat{s}^C \in L$, P the set of all relative sub lattices of L. By an outer measure Ɲ* we mean an extended real valued set function defined on the set P of all relative sub lattices, H⊂L satisfying the following properties.

(1) $\text{Ɲ}^*(\phi)=0$

(2) $Ƨ_1 \subseteq Ƨ_2 \Rightarrow \text{Ɲ}^*(Ƨ_1) \le \text{Ɲ}^*(Ƨ_2)$

[Monotonicity]

(3) $\hat{s} \subseteq \bigvee_{i=1}^{\infty} \hat{s}_i \Rightarrow \text{Ɲ}^*(\hat{s}) \le \sum_{i=1}^{\infty} \text{Ɲ}^*(\hat{s}_i)$

[Countable sub-additive]

Note3.1. From the definition Ɲ* is finitely sub-additive.

Definition3.2. [4] A relative sub lattice ŝ is said to be measurable with respect to Ɲ* if $\text{Ɲ}^*(Ƨ) = \text{Ɲ}^*(Ƨ \wedge \hat{s}) + \text{Ɲ}^*(Ƨ \wedge \hat{s}^C)$ for every relative sub lattice Ƨ, where $\hat{s}^C = X - \hat{s}$.

Note 3.2. For any Ƨ in P, we can write $Ƨ = (Ƨ \wedge \hat{s}) \vee (Ƨ \wedge \hat{s}^C) \Rightarrow \text{Ɲ}^*(Ƨ) \le \text{Ɲ}^*(Ƨ \wedge \hat{s}) + \text{Ɲ}^*(Ƨ \wedge \hat{s}^C)$. So to show that E is measurable, it is enough to show $\text{Ɲ}^*(Ƨ) \ge \text{Ɲ}^*(Ƨ \wedge \hat{s}) + \text{Ɲ}^*(Ƨ \wedge \hat{s}^C)$ for every Ƨ∈P.

Definition3.3. By a measure on an algebra Q of relative sub lattices in a C.B.L $L \subset \mathbf{P}(X)$ we mean it follows measure on an algebra Q**[4]**

Example3.1. Let D = {Δ/Δ⊆□ is an open set in [0, 1]}. Clearly Δ is a relative sub lattice under usual ordering on □.Now m∈Δ, n∈Δ ⇒ m∨n = m or n. So m∨n ∈Δ and m∧n∈Δ

If $\{\Delta_l\}$ is a sequence of open sets then so is $\vee \Delta_l$

Clearly $\phi \in D$
Now $\Delta' = \{1 - m / m \in \Delta\}$ is open set
Δ is open, $\Rightarrow -\Delta$ is open.
$\Rightarrow 1 - \Delta$ is open set
$\Rightarrow \Delta'$ is open set and $\Delta' \subseteq [0, 1]$
$\Rightarrow \Delta' \in D$ whenever $\Delta \in D$.
Let Ɲ be the Lebesgue measure on□, $\Rightarrow$ Ɲ$(\phi) = 0$. If $\{\Delta_l\}$ is a disjoint collection in D then
Ɲ$(\vee \Delta_l) = \sum$ Ɲ(Δ_l) from the Lebesgue theory.

Definition3.4. [4] A lattice measure space (X, $L(\mathscr{B})$, Ɲ) is complete if $L(\mathscr{B})$possess all relative sub lattices of members of $L(\mathscr{B})$of measure zero. That is, if $B \in L(\mathscr{B})$ and Ɲ$(B) = 0$ and $S \subseteq B \Rightarrow S \in L(\mathscr{B})$.
Here the author proceeds with a measure on an algebra Q of relative sub lattices and later he expands this to a measure defined on an sigma- algebra $L(\mathscr{B})$ containing Q. First, we define outer measure on P.
Definition3.5. Let Ɲ be a measure on an algebra Q of relative sub lattices, for any relative sub lattice ŝ, we define

Ɲ*(ŝ) = Inf $\{\sum_{i=1}^{\infty}$ Ɲ(S_i) / $\{S_i\}$ ranges over all sequence from Q $\ni$ ŝ $\subseteq \bigvee_{i=1}^{\infty} S_i\}$ Ɲ* is called the outer measure generated by Ɲ.
Note3.4. Clearly Ɲ*(ŝ) ≥ 0 $\forall$ ŝ $\in$P
Definition3.6. [4] Ɲ* is said to be regular if
given any relative sub lattice ŝ of X and any $\mathcal{E}>0$
there is a Ɲ* measurable relative sub lattice S with ŝ $\subseteq$ S and Ɲ*(S) $\leq$ Ɲ*(ŝ) + $\mathcal{E}$.

Note3.5. A relative sub lattice H in P is Ɲ* measurable if Ɲ*(ŝ) = Ɲ*(ŝ $\wedge$H) + Ɲ*(ŝ $\wedge H^c$) $\forall$ ŝ $\in$P.

Ɲ* MEASURABLE RELATIVE SUB LATTICE

Result4.1. [1] For S $\in$L we have Ɲ*(S) = Ɲ(S)
Result4.2. [2] If (L, L, Ɲ) is a lattice measure space then we can find a complete measure space (L, L_0, $Ɲ_0$) $\ni$
(i) $L \subseteq L_0$
(ii) ŝ $\in$**L** $\Rightarrow$ Ɲ(ŝ) = $Ɲ_0$(ŝ)
(iii) ŝ $\in$L $\Leftrightarrow$ ŝ = S $\vee$ B, where B$\in$ L and S$\subseteq$C, C$\in$ L, Ɲ(C) = 0
Theorem4.1. $L(\mathscr{B})$ in P is a sigma-algebra. If $\overline{Ɲ}$ is Ɲ* restricted to $L(\mathscr{B})$, then $\overline{Ɲ}$ is complete measure on $L(\mathscr{B})$.
Proof. Claim. $L(\mathscr{B})$ is a sigma-algebra.
Part – I: For any set Ƨ,
(i) Ɲ*(Ƨ$\wedge\phi$) + Ɲ*(Ƨ$\wedge\phi^c$)
= Ɲ*(ϕ) + Ɲ*(Ƨ) = Ɲ*(Ƨ)
$\Rightarrow \phi$ is Ɲ* measurable.
$\Rightarrow \phi \in L(\mathscr{B})$
Therefore $L(\mathscr{B}) \neq \phi$
(ii) We show that ŝ $\in L(\mathscr{B}) \Rightarrow$ ŝ^C $\in L(\mathscr{B})$
Let E $\in L(\mathscr{B})$.
Then Ɲ*(Ƨ) = Ɲ*(Ƨ$\wedge$ ŝ) + Ɲ*(Ƨ$\wedge$ ŝ^C) for every relative sub lattice Ƨ.
$\Rightarrow$ Ɲ*(Ƨ) = Ɲ(Ƨ$\wedge$ ŝ^C) + Ɲ*(Ƨ$\wedge$ ŝ) for every set Ƨ.
$\Rightarrow$ ŝ^C is Ɲ* measurable.
$\Rightarrow$ ŝ^C $\in L(\mathscr{B})$
(iii) Let ŝ$_1$, ŝ$_2$ $\in L(\mathscr{B})$, $\Rightarrow$ ŝ$_1 \vee$ ŝ$_2$ $\in L(\mathscr{B})$.
Now we show ŝ$_1 \vee$ ŝ$_2$ $\in L(\mathscr{B})$.
That is Ɲ*(Ƨ) = Ɲ*[Ƨ$\wedge$ (ŝ$_1 \vee$ ŝ$_2$)] + Ɲ*[Ƨ$\wedge$ (ŝ$_1 \vee$ ŝ$_2$)^C] for E$\in$P
Clearly Ɲ*(Ƨ) $\leq$ Ɲ*[Ƨ$\wedge$ (ŝ$_1 \vee$ ŝ$_2$)] + Ɲ*[Ƨ$\wedge$ (ŝ$_1^C \wedge$ ŝ$_2^C$)]
Consider
Ɲ*[Ƨ$\wedge$ (ŝ$_1 \vee$ ŝ$_2$)] + Ɲ*[Ƨ$\wedge$ (ŝ$_1^C \wedge$ ŝ$_2^C$)]

$= Ɲ^*\{Ƨ \wedge [\hat{s}_1 \vee (\hat{s}_2 - \hat{s}_1)]\} +$
$Ɲ^*[Ƨ \wedge (\hat{s}_1^C \wedge \hat{s}_2^C)]$
$= Ɲ^*\{(Ƨ \wedge \hat{s}_1) \vee [Ƨ \wedge (\hat{s}_2 \wedge \hat{s}_1^C)]\} +$
$Ɲ^*[Ƨ \wedge (\hat{s}_1^C \wedge \hat{s}_2^C)]$
$\leq Ɲ^*(Ƨ \wedge \hat{s}_1) + Ɲ^*[Ƨ \wedge (\hat{s}_2 \wedge \hat{s}_1^C)] +$
$Ɲ^*[Ƨ \wedge (\hat{s}_1^C \wedge \hat{s}_2^C)]$
$= Ɲ^*(Ƨ \wedge \hat{s}_1) + Ɲ^*[(Ƨ \wedge \hat{s}_1^C) \wedge \hat{s}_2]\} +$
$Ɲ^*[(Ƨ \wedge \hat{s}_1^C) \wedge \hat{s}_2^C]$
$= Ɲ^*(Ƨ \wedge \hat{s}_1) + Ɲ^*(Ƨ \wedge \hat{s}_1^C)$
[Since $\hat{s}_2$ is $Ɲ^*$ measurable]
$= Ɲ^*(Ƨ)$

Therefore
$Ɲ^*(Ƨ) = Ɲ^*[Ƨ \wedge (\hat{s}_1 \vee \hat{s}_2)] + Ɲ^*[Ƨ \wedge (\hat{s}_1 \vee \hat{s}_2)^C]$
$\Rightarrow \hat{s}_1 \vee \hat{s}_2$ is measurable.
$\Rightarrow \hat{s}_1 \vee \hat{s}_2 \in L(\mathscr{B})$
So, the join of two measurable relative sub lattices is measurable and by induction the join of any finite number of measurable relative sub lattices is measurable.
Therefore $L(\mathscr{B})$ is algebra of relative sub Lattices.
Let $\hat{s} = \bigvee_i \hat{s}_i$, where $\{\hat{s}_i\}$ is disjoint sequence of measurable relative sub lattices.

Write $G_n = \bigvee_{i=1}^{n} \hat{s}_i$

Then G_n is measurable $\forall$ n.
Claim. $Ɲ^*(Ƨ) \geq Ɲ^*(Ƨ \wedge \hat{s}) + Ɲ^*(Ƨ \wedge \hat{s}^C)$
Consider

$Ɲ^*(Ƨ) = Ɲ^*(Ƨ \wedge G_n) + Ɲ^*(Ƨ \wedge G_n^c)$
[Since G_n is measurable]
$\geq Ɲ^*(Ƨ \wedge G_n) + Ɲ^*(Ƨ \wedge \hat{s}^C)$
[Since $G_n \subseteq \hat{s} \Rightarrow \hat{s}^C \subseteq G_n^c$ is measurable]
$= Ɲ^*[(Ƨ \wedge G_n) \wedge \hat{s}_n)] + Ɲ^*[(Ƨ \wedge G_n) \wedge \hat{s}_n^C)] +$
$Ɲ^*(Ƨ \wedge E^c)$
[Since E_n is measurable]

$= Ɲ^*(Ƨ \wedge \hat{s}_n) + Ɲ^*(Ƨ \wedge G_{n-1}) + Ɲ^*(Ƨ \wedge \hat{s}^C)$

[Since $\hat{s}_n \subseteq G_n$. $\Rightarrow G_n \wedge \hat{s}_n = \hat{s}_n$, $G_n \wedge \hat{s}_n^C = G_{n-1}$]

$= Ɲ^*(Ƨ \wedge \hat{s}_n) + Ɲ^*[Ƨ \wedge (\bigvee_{i=1}^{n-1} \hat{s}_i)] + Ɲ^*(Ƨ \wedge \hat{s}^C)$

$= Ɲ^*(Ƨ \wedge \hat{s}_n) + \sum_{i=1}^{n-1} Ɲ^*(Ƨ \wedge \hat{s}_i) + Ɲ^*(Ƨ \wedge \hat{s}^C)$

[Since $Ɲ^*(Ƨ \wedge G_{n-1}) = \sum_{i=1}^{n-1} Ɲ^*(Ƨ \wedge \hat{s}_i)$]

$= \sum_{i=1}^{n} Ɲ^*(Ƨ \wedge \hat{s}_i) + Ɲ^*(Ƨ \wedge \hat{s}^C)$

Therefore

$Ɲ^*(Ƨ) \geq Ɲ^*(Ƨ \wedge \hat{s}^C) + \sum_{i=1}^{n} Ɲ^*(Ƨ \wedge \hat{s}_i) \; \forall$ n

$$\Rightarrow Ɲ^*(Ƨ) \geq Ɲ^*(Ƨ \wedge \hat{s}^C) + \sum_{i=1}^{\infty} Ɲ^*(Ƨ \wedge \hat{s}_i)$$

$$\geq Ɲ^*(Ƨ \wedge \hat{s}^C) + Ɲ^*(Ƨ \wedge \hat{s})$$

[Since $Ƨ \wedge \hat{s} = \bigvee_{i=1}^{\infty} Ƨ \wedge \hat{s}_i$ and

$$Ɲ^*(Ƨ \wedge \hat{s}) = \sum_{i=1}^{\infty} Ɲ^*(Ƨ \wedge \hat{s}_i)]$$

Therefore $Ɲ^*(Ƨ) \geq Ɲ^*(Ƨ \wedge \hat{s}^C) + Ɲ^*(Ƨ \wedge \hat{s})$

Therefore, $\hat{s}$ is measurable.

Since the join of any sequence of relative sub lattices in algebra Q can be replaced by disjoint join of relative sub lattices in the algebra, it follows that $L(\mathscr{B})$ is a sigma-algebra.

Part – II: Suppose $\overline{Ɲ} = Ɲ^*/L(\mathscr{B})$

Claim. $\overline{Ɲ}$ is a complete measure on $L(\mathscr{B})$.

(i) By result 4.1 and result 4.2 (ii), we have

$\overline{Ɲ}(\hat{s}) = Ɲ^*(\hat{s}) \ \forall\ \hat{s} \in L(\mathscr{B})$

$\Rightarrow \overline{Ɲ}(\phi) = 0$

$\Rightarrow Ɲ^*(\phi) = 0$

Also $\overline{Ɲ}$ is a non – negative function defined on $L(\mathscr{B})$ which is the class of a $Ɲ^*$ measurable relative sub lattices $L(\mathscr{B})$.

(i) First, we show that $\overline{Ɲ}$ is finitely-additive.

Let $\hat{s}_1, \hat{s}_2 \in L(\mathscr{B})$ and $\hat{s}_1 \wedge \hat{s}_2 = \phi$

Consider $\overline{Ɲ}(\hat{s}_1 \vee \hat{s}_2) = Ɲ^*(\hat{s}_1 \vee \hat{s}_2)$

$= Ɲ^*[(\hat{s}_1 \vee \hat{s}_2) \wedge \hat{s}_1] + Ɲ^*[(\hat{s}_1 \vee \hat{s}_2) \wedge \hat{s}_1^C]$

[Since E_1 is $Ɲ^*$ measurable]

$= Ɲ^*(\hat{s}_1) + Ɲ^*(\hat{s}_2)$

[By absorption law and distributive law]

$= \overline{Ɲ}(\hat{s}_1) + \overline{Ɲ}(\hat{s}_2)$

Therefore $\overline{Ɲ}(\hat{s}_1 \vee \hat{s}_2) = \overline{Ɲ}(\hat{s}_1) + \overline{Ɲ}(\hat{s}_2)$

So this holds for n = 2

Suppose, this is true for n – 1

Take $\{\hat{s}_1, \hat{s}_2, \ldots\ldots\ldots \hat{s}_{n-1}, \hat{s}_n\}$ be a disjoint collection of relative sub lattices in $L(\mathscr{B})$.

Then $\bigvee_{i=1}^{n-1} \hat{s}_i$ and $\hat{s}_n$ are disjoint relative sub lattices in $L(\mathscr{B})$

So $\overline{Ɲ}(\bigvee_{i=1}^{n} \hat{s}_i) = \overline{Ɲ}[(\bigvee_{i=1}^{n-1} \hat{s}_i) \vee \hat{s}_n]$

$$= \overline{Ɲ}[(\bigvee_{i=1}^{n-1} \hat{s}_i)] + \overline{Ɲ}(\hat{s}_n)$$

$$= \sum_{i=1}^{n-1} \overline{Ɲ}(\hat{s}_i) + \overline{Ɲ}(\hat{s}_n) = \sum_{i=1}^{n} \overline{Ɲ}(\hat{s}_i)$$

Hence this holds for n.

By induction this holds $\forall$ n.

Therefore $\overline{Ɲ}$ is finitely-additive

(iii) Let $\{\hat{s}_i\}$ be a disjoint sequence of measurable relative sub lattices and

$$\hat{s} = \bigvee_{i=1}^{\infty} \hat{s}_i$$

Then $\bigvee_{i=1}^{n} \hat{s}_i \subseteq \hat{s}$ this is true $\forall$ n

$\Rightarrow \overline{N}(\hat{s}) \geq \overline{N}(\bigvee_{i=1}^{n} \hat{s}_i) = \sum_{i=1}^{n} \overline{N}(\hat{s}_i) \; \forall \; n.$

$\Rightarrow \overline{N}(\hat{s}) \geq \sum_{i=1}^{\infty} \overline{N}(\hat{s}_i)$ ------------ (1)

But $\overline{N}(\hat{s}) = N^*(\hat{s}) = N^*(\bigvee_{i=1}^{\infty} \hat{s}_i) \leq \sum_{i=1}^{\infty} N^*(\hat{s}_i)$

$= \sum_{i=1}^{\infty} \overline{N}(\hat{s}_i)$

$\Rightarrow \overline{N}(\hat{s}) \leq \sum_{i=1}^{\infty} \overline{N}(\hat{s}_i)$ -------------- (2)

From (1) and (2),

$\overline{N}(\hat{s}) = \sum_{i=1}^{\infty} \overline{N}(\hat{s}_i)$

Therefore $\overline{N}$ is measurable on $L(\mathscr{B})$.

Now we claim that $\overline{N}$ is complete.

Let B be any measurable relative sub lattice $\ni \overline{N}(B) = 0$.

Let $\mathcal{S} \subseteq B$.

Claim. $\mathcal{S} \in L(\mathscr{B})$.

Since $\mathcal{S} \subseteq B$, we have $N^*(\mathcal{S}) \leq N^*(B) = \overline{N}(B) = 0$.

$\Rightarrow N^*(\mathcal{S}) = 0$.

For any set C, $C \wedge \mathcal{S} \subseteq \mathcal{S}$, $C \wedge \mathcal{S}^C \subseteq C$

$\Rightarrow N^*(C \wedge \mathcal{S}) \leq N^*(\mathcal{S}) = 0$

$\Rightarrow N^*(C \wedge \mathcal{S}) = 0$ and $N^*(C \wedge \mathcal{S}^C) \leq N^*(C)$

That is $N^*(C) \geq N^*(C \wedge \mathcal{S}^C) + 0$

$\Rightarrow N^*(C) \geq N^*(C \wedge \mathcal{S}^C) + N^*(C \wedge \mathcal{S}) \; \forall \; C$

$\Rightarrow \mathcal{S}$ is measurable.

$\Rightarrow \mathcal{S} \in L(\mathscr{B})$.

So $L(\mathscr{B})$ contains all relative sub lattices of measure zero.

Therefore $\overline{N}$ is complete measure.

Hence the theorem

Lemma4.1. Let Q be algebra of relative sub lattices and N be a measure on an algebra $L(\mathscr{B})$. If $\{S_i\}$ is any sequence of relative sub lattices in Q $\ni S \subseteq \bigvee_{i=1}^{\infty} S_i$ then $N(S) \leq \sum_{i=1}^{\infty} N(S_i)$.

Proof. Write $B_n = S \wedge S_n \wedge (\bigvee_{i=1}^{n-1} S_i)^C$

Then $B_n \in Q$ and $B_n \subseteq S_n$, also $B_n \subseteq S_m^C$ for $m < n$,

For any $n > m$,

$B_n \wedge B_m \subseteq S_m^C \wedge B_m \subseteq S_m^C \wedge S_m = \phi$.

Therefore $\{B_n\}$ is a sequence of disjoint relative sub lattices from S.

Now we show $S = \bigvee_n B_n$.

By definition $B_n \subseteq S \wedge S_n \; \forall \; n$.

$\Rightarrow \bigvee_{n=1}^{\infty} B_n \subseteq \bigvee_{n=1}^{\infty} (S \wedge S_n) = S$ [since $S \subseteq \bigvee_{i=1}^{\infty} S_i \Rightarrow S = S \wedge (\bigvee_{i=1}^{\infty} S_i) = \bigvee_{i=1}^{\infty} (S \wedge S_i)$]

Let $x \in S \subseteq \bigvee_n S_n$.

$\Rightarrow x \in S_n$ for some n

Let n be the smallest positive integer $\ni x \in S_n$

$\Rightarrow x \notin S_i \ \forall\ i < n$

$\Rightarrow x \notin (\bigvee_{i=1}^{n-1} S_i)$

$\Rightarrow x \in (S \wedge S_n) - (\bigvee_{i=1}^{n-1} S_i) = B_n$

Therefore $S \subseteq \bigvee_n B_n$

Therefore $S = \bigvee_n B_n$ and $\{B_n\}$ is a sequence of disjoint relative sub lattices in Q. Then by definition of measure on algebra we have

$N(\bigvee_n B_n) = \sum_n N(B_n)$

$\Rightarrow N(S) = \sum_n N(B_n) \leq \sum_n N(S_n)$

[Since $B_n \subseteq S_n$]

Hence the lemma

Corollary4.1. Let N be a measure on an algebra Q of relative sub lattices. If $S \in Q$ then $N^*(S) = N(S)$.

Proof. Let $S \in Q$

Since $S \subseteq S$, by definition 3.1, $N^*(S) \leq N(S)$

[Since $N^*(S)$ is infimum]

By the lemma 4.1, $N(S) \leq \sum_{i=1}^{\infty} N(S_i)$ for any sequence $\{S_i\} \ni S \subseteq \bigvee_{i=1}^{\infty} S_i$

$\Rightarrow N(S) \leq N^*(S)$ [By definition 3.1]

$\Rightarrow N(S) = N^*(S)$

Hence the corollary

Lemma4.2. N* is an outer measure.

Proof. Clearly N* is a non-negative extended real valued function.

(i) $N^*(\phi) = 0$ is clear [since $\phi \in L(\mathscr{B})$]

(ii) suppose $S \subseteq B$

Let $\{B_i\}$ where i = 1 to ∞ be a sequence of relative sub lattices in $Q \ni B \subseteq \bigvee_{i=1}^{\infty} B_i$

$\Rightarrow S \subseteq B \subseteq \bigvee_{i=1}^{\infty} B_i$

$\Rightarrow S \subseteq \bigvee_{i=1}^{\infty} B_i$

$\Rightarrow N^*(S) \leq \sum_{i=1}^{\infty} N(B_i)$ [By definition 3.1]

$\Rightarrow N^*(S) \leq \text{Inf}\, \{\sum_{i=1}^{\infty} N(B_i) / B \subset \bigvee_{i=1}^{\infty} B_i$, where $\{B_i\}$ is a sequence of relative sub lattice in Q$\}$.

$\Rightarrow N^*(S) \leq N^*(B)$

Therefore N* is monotone.

(iii) Suppose $\hat{s} \subseteq \bigvee_{i=1}^{\infty} \hat{s}_i$

Claim. $N^*(\hat{s}) \leq \sum_{i=1}^{\infty} N(\hat{s}_i)$

If $\mathcal{N}^*(\hat{s}_i) = \infty$ for some i, then there is nothing to prove.
Suppose $\mathcal{N}^*(\hat{s}_i) < \infty, \forall i$.

Let $\mathcal{E} > 0$.
Now $\mathcal{N}^*(\hat{s}_i) + \frac{\varepsilon}{2^i}$ is not a lower bound of the set

$\{\sum_{j=1}^{\infty} \mathcal{N}(S_{ij}) / \hat{s}_i \subseteq \bigvee_{j=1}^{\infty} S_{ij}$, where $\{S_{ij}\}$ is sequence of relative sub lattices in Q$\}$. $\Rightarrow \exists \{S_{ij}\}$ where j = 1 to ∞ of relative

sub lattice $\ni \hat{s}_i \subseteq \bigvee_{j=1}^{\infty} S_{ij}$ and $\sum_{j=1}^{\infty} \mathcal{N}(S_{ij}) < \mathcal{N}^*(\hat{s}_i) + \frac{\varepsilon}{2^i}$.

This is true for each i.

Now $\hat{s} \subseteq \bigvee_{i=1}^{\infty} \hat{s}_i = \bigvee_{i=1}^{\infty} \bigvee_{j=1}^{\infty} S_{ij} = \bigvee_{i,j} S_{ij}$

So by definition 3.1, we have

$\mathcal{N}^*(\hat{s}) \leq \sum_{i,j} \mathcal{N}(S_{ij}) = \sum_i [\sum_j \mathcal{N}(S_{ij})]$

$< \sum_i [\mathcal{N}^*(\hat{s}_i) + \frac{\varepsilon}{2^i}]$

$= \sum_i \mathcal{N}^*(\hat{s}_i) + \sum_i \frac{\varepsilon}{2^i}$

$= \sum_i \mathcal{N}^*(\hat{s}_i) + \mathcal{E}$

[Since $\sum_i \frac{1}{2^i} = 1$]

Since $\mathcal{E}>0$ is arbitrary, we have

$\mathcal{N}^*(\hat{s}) \leq \sum_{i=1}^{\infty} \mathcal{N}^*(\hat{s}_i)$

Therefore $\mathcal{N}^*$ is countable sub-additive.
Hence $\mathcal{N}^*$ is an outer measure.
Hence the lemma

Lemma4.3. Let Q be algebra of relative sub lattices and $\mathcal{N}$ be a measure on the algebra Q. If $\mathcal{S} \in Q$ then $\mathcal{S}$ is measurable with respect to $\mathcal{N}^*$.

Proof. Let $\hat{s}$ be any arbitrary set.
Claim. $\mathcal{N}^*(\hat{s}) = \mathcal{N}^*(\hat{s} \wedge \mathcal{S}) + \mathcal{N}^*(\hat{s} \wedge \mathcal{S}^C)$

Clearly $\mathcal{N}^*(\hat{s}) \leq \mathcal{N}^*(\hat{s} \wedge \mathcal{S}) + \mathcal{N}^*(\hat{s} \wedge \mathcal{S}^C)$
Now it remains to show,
$\mathcal{N}^*(\hat{s}) \geq \mathcal{N}^*(\hat{s} \wedge \mathcal{S}) + \mathcal{N}^*(\hat{s} \wedge \mathcal{S}^C)$
If $\mathcal{N}^*(\hat{s}) = \infty$ then there is nothing to prove
Suppose $\mathcal{N}^*(\hat{s}) < \infty$ and let $\mathcal{E} > 0$
By definition of $\mathcal{N}^*(\hat{s})$, $\mathcal{N}^*(\hat{s}) + \mathcal{E}$ is not lower bound of set

$\{\sum_{i=1}^{\infty} \mathcal{N}(\mathcal{S}_i) / \hat{s} \subseteq \bigvee_{i=1}^{\infty} \mathcal{S}_i\}$.

$\Rightarrow \exists$ a sequence $\{S_i\}$ of relative sub lattices in Q

$\ni \hat{s} \subseteq \bigvee_{i=1}^{\infty} \mathcal{S}_i$ and $\sum_{i=1}^{\infty} \mathcal{N}(\mathcal{S}_i) < \mathcal{N}^*(\hat{s}) + \mathcal{E}$.

For each i,
We can write $\mathcal{S}_i = \mathcal{S}_i \wedge X$

$= Ƹ_i \wedge (Ƹ \vee Ƹ^C) = (Ƹ \wedge Ƹ_i) \vee (Ƹ_i \wedge Ƹ^C)$

$\Rightarrow Ɲ(Ƹ_i) = Ɲ(Ƹ \wedge Ƹ_i) + Ɲ(Ƹ_i \wedge Ƹ^C) \ \forall\ i$ --------- (1)

[By definition 3.4]

Consider

$ŝ \wedge Ƹ \subseteq \bigvee_{i=1}^{\infty} (Ƹ \wedge Ƹ_i)$ and $ŝ \wedge Ƹ^C \subseteq \bigvee_{i=1}^{\infty} (Ƹ_i \wedge Ƹ^C)$

Then $Ɲ^*(ŝ \wedge Ƹ) \leq \sum_{i=1}^{\infty} Ɲ(Ƹ \wedge Ƹ_i)$ and $Ɲ^*(ŝ \wedge Ƹ^C) \leq \sum_{i=1}^{\infty} Ɲ(Ƹ_i \wedge Ƹ^C)$ [By definition 3.1]

Consider

$Ɲ^*(ŝ \wedge Ƹ) + Ɲ^*(ŝ \wedge Ƹ^C)$

$\leq \sum_{i=1}^{\infty} Ɲ(Ƹ \wedge Ƹ_i) + \sum_{i=1}^{\infty} Ɲ(Ƹ_i \wedge Ƹ^C)$

$= \sum_{i=1}^{\infty} [Ɲ(Ƹ \wedge Ƹ_i) + Ɲ(Ƹ_i \wedge Ƹ^C)] = \sum_{i=1}^{\infty} Ɲ(Ƹ_i)$ [from (1)] $< Ɲ^*(ŝ) + Ɛ$

Since $Ɛ > 0$ is arbitrary, we have

$Ɲ^*(ŝ) \geq Ɲ^*(ŝ \wedge Ƹ) + Ɲ^*(ŝ \wedge Ƹ^C) \ \forall\ ŝ$

$\Rightarrow$ With respect to $Ɲ^*$, $Ƹ$ is measurable.

Note4.1.outer measure induced by Ɲ also said to be the outer measure $Ɲ^*$

OUTER MEASURE GENERATED BY Ɲ

Note5.1. Let Q be algebra of relative sub lattices. As mentioned earlier we use Q_σ to denote those relative sub lattices that are countable join of relative sub lattices of Q and also we use $Q_{\sigma\delta}$ to denote those relative sub lattices that are countable intersections of relative sub lattices in Q_σ.

Clearly $Q \subseteq Q_\sigma \subseteq Q_{\sigma\delta}$

Theorem5.1. Let Ɲ be a measure on an algebra Q of relative sub lattices in P of $L \subset$ **P**(X) and $Ɲ^*$ be the outer measure induced by Ɲ and ŝ be any relative sub lattice. Then for every $Ɛ > 0$ there is a relative sub lattice $S \in Q_\sigma$ with $ŝ \subseteq A$ and $Ɲ^*(S) \leq Ɲ^*(ŝ) + Ɛ$. There is also a relative sub lattice $B \in Q_{\sigma\delta}$ with $ŝ \subseteq S$ and $Ɲ^*(ŝ) = Ɲ^*(B)$.

Proof.

Part – I: Let $Ɛ > 0$

Then by definition of $Ɲ^*(ŝ)$, we have

$Ɲ^*(ŝ) \leq \text{Inf}\ \{\sum_{i=1}^{\infty} Ɲ(S_i) / ŝ \subset \bigvee_{i=1}^{\infty} S_i$, where $\{S_i\}$ is a sequence of relative sub lattice in Q}.

Then $Ɲ^*(ŝ) + Ɛ$ is not a lower bound of the set

$\{\sum_{i=1}^{\infty} Ɲ(S_i) / ŝ \subset \bigvee_{i=1}^{\infty} S_i$, where $\{S_i\}$ is a sequence of relative sub lattice in Q}.

$\Rightarrow \exists$ a sequence $\{S_i\}$ of relative sub Lattices in Q $\ni ŝ \subseteq \bigvee_i S_i$ and $\sum_{i=1}^{\infty} Ɲ(S_i) < Ɲ^*(ŝ) + Ɛ$.

Write $S = \bigvee_i S_i$

Then $S \subseteq Q_\sigma$ and $ŝ \subseteq S$

Also $Ɲ^*(S) \leq Ɲ^*(\bigvee_i S_i)$

$\leq \sum_{i=1}^{\infty} Ɲ^*(S_i)$

[Since $Ɲ^*$ is countable sub-additive]

$$= \sum_{i=1}^{\infty} N(S_i)$$

[By corollary 4.1]

$\leq N^*(E) + \varepsilon$

Therefore, for any $\varepsilon > 0$, there is a relative sub lattice S in $Q_\sigma \ni \hat{s} \subseteq S$ and $N^*(S) \leq N^*(\hat{s}) + \varepsilon$.

Part – II: By part – I, for each positive integer n there is a relative sub lattice $S_n \in Q_\sigma \ni \hat{s} \subseteq A_n$ and $N^*(S) \leq N^*(\hat{s}) + \frac{1}{n}\ \forall n$.

Write $B = \bigwedge_{i=1}^{\infty} S_n$.

Then $B \in Q_{\sigma\delta}$ and $\hat{s} \subseteq B$

Now $N^*(B) = N^*(\bigwedge_n S_n) \leq N^*(S_n)$

$\leq N^*(\hat{s}) + \frac{1}{n}$, this is true $\forall$ n.

$\Rightarrow N^*(B) \leq N^*(\hat{s})$

Also, $N^*(\hat{s}) \leq N^*(B)$ since $\hat{s} \subseteq B$

So $N^*(B) = N^*(\hat{s})$

Therefore, there is also a relative sub lattice B in $Q_{\sigma\delta}$ and $\hat{s} \subseteq B$ and

$N^*(\hat{s}) = N^*(B)$.

Hence the theorem

Corollary5.1. The outer measure induced by a measure on algebra Q is a regular outer measure.

Proof. By lemma 4.3 and Theorem 5.1.

Theorem5.2. Let N be a sigma-finite measure on an algebra Q and let N^* be the outer measure induced by N. A relative sub lattice E is N^* measurable $\Leftrightarrow$ $\hat{s}$ is the proper difference $S \sim B$ of a relative sub lattice A in $Q_{\sigma\delta}$ and a relative sub lattice B with $N^*(B) = 0$. Each relative sub lattice B with $N^*(B) = 0$ is contained in a relative sub lattice C in $Q_{\sigma\delta}$ with $N^*(C) = 0$.

Proof.

Write $L(\mathscr{B})$ = the class of all N^* measurable relative sub lattices. Then $L(\mathscr{B})$ is a sigma-algebra and $Q \subseteq L(\mathscr{B})$ (by lemma 4.3, each relative sub lattice in Q is measurable).

$\Rightarrow Q_{\sigma\delta} \subseteq L(\mathscr{B})$

[By theorem 4.1, $L(\mathscr{B})$ is a sigma-algebra]

Write $\overline{N} = N^*/L(\mathscr{B})$.

Then $\overline{N}$ is a complete measure

[By theorem 4.1]

Suppose $\hat{s} = S \sim B$, where $S \in Q_{\sigma\delta}$, B is $\ni$ $N^*(B) = 0$.

Since $S \in Q_{\sigma\delta}$, we have $S \in L(\mathscr{B})$.

Since $N^*(B) = 0$, we have $B \in L(\mathscr{B})$.

Therefore $S, B \in L(\mathscr{B})$

$\Rightarrow E = S \sim B \in L(\mathscr{B})$.

$\Rightarrow$ E is N^* measurable.

Conversely, suppose that $\hat{s}$ is N^* measurable, that is, $E \in L(\mathscr{B})$.

Part – I: since N is sigma-finite, by definition $\exists$ a sequence of relative sub lattices $\{X_i\}$ in $Q \ni X = \bigvee_i X_i$ and $N(X_i) < \infty, \forall$ i.

We can suppose that X_i's are disjoint.

Now $\hat{s} = \hat{s} \wedge X = \hat{s} \wedge (\bigvee_i X_i) = \bigvee_i (\hat{s} \wedge X_i)$

$= \bigvee_i \hat{s}_i$

Where $\hat{s}_i = \hat{s} \wedge X_i \in L(\mathscr{B})$

[Since $\hat{s} \subseteq L(\mathscr{B})$, $X_i \in Q \subseteq L(\mathscr{B})$]

Consider $\hat{s}_i = X_i \wedge E$
Then $\overline{\mathrm{N}}(\hat{s}_i) = \mathrm{N}^*(\hat{s}_i) \le \mathrm{N}^*(X_i) < \infty$.
[Since $\overline{\mathrm{N}} = \mathrm{N}^*/L(\mathscr{B})$, $\hat{s}_i \in L(\mathscr{B})$]

Part – II: Now we show that for each positive integer, there is a set $S_n \in Q_\sigma \ni \overline{\mathrm{N}}(S_n - \hat{s}) \le \frac{1}{n}$ $\forall$ n.

By Theorem 5.1, for each positive integer n, there exist a relative sub lattice $S_{n,i}$ in $Q_\sigma \ni \mathrm{N}^*(S_{n,i}) \le \mathrm{N}^*(\hat{s}_i) + \frac{1}{n\,2^i}$

and $\hat{s}_i \subseteq S_{n,i}$ $\forall$i.

Put $S_n = \bigvee_{i=1}^{\infty} S_{n,i}$

Then $\hat{s} = \bigvee_i \hat{s}_i \subseteq \bigvee_i S_{n,i} = S_n$, $\forall$ n and $S_n \in Q_\sigma$

Also $S_n - \hat{s} \subseteq \bigvee_i (S_{n,i} - \hat{s}_i)$

[Since $S_n - \hat{s} = (\bigvee_i S_{n,i}) - \hat{s} \subseteq \bigvee_i (S_{n,i} - \hat{s}_i)$]

$$\overline{\mathrm{N}}(S_n - \hat{s}) \le \sum_{i=1}^{\infty} \overline{\mathrm{N}}(S_{n,i} - \hat{s}_i)$$

[Here $\overline{\mathrm{N}}$ and N^* are same on $L(\mathscr{B})$]

$$\le \sum_{i=1}^{\infty} [\overline{\mathrm{N}}(S_{n,i}) - \overline{\mathrm{N}}(\hat{s}_i)]$$

[Since $\hat{s} \subseteq S_{n,i}$. $\Rightarrow S_{n,i} = \hat{s}_i \vee (S_{n,i} - \hat{s}_i)$
$\Rightarrow \overline{\mathrm{N}}(S_{n,i}) = \overline{\mathrm{N}}(\hat{s}_i) + \overline{\mathrm{N}}(S_{n,i} - \hat{s}_i)$]

$$\le \sum_{i=1}^{\infty} \frac{1}{n\,2^i} = \frac{1}{n} \sum_{i=1}^{\infty} \frac{1}{2^i}$$

$$= \frac{1}{n} \qquad \left[\text{since } \sum_{i=1}^{n} \frac{1}{2^i} = 1\right]$$

Therefore $\overline{\mathrm{N}}(S_n \sim \hat{s}) \le \frac{1}{n}$ $\forall$ n

Write $S = \bigwedge_{n=1}^{\infty} S_n$

Then $S \in Q_{\sigma\delta}$ [Since $S_n \in Q_\sigma$]
We write $\hat{s} = S \sim (S \sim \hat{s})$, where S is relative sub lattice in $Q_{\sigma\delta}$ and $\mathrm{N}^*(S \sim \hat{s}) = 0$

[Since $S \sim E \subseteq S_n \sim E$ $\forall$ n

$\Rightarrow 0 \le \overline{\mathrm{N}}(S \sim \hat{s}) \le \overline{\mathrm{N}}(S_n \sim \hat{s}) \le \frac{1}{n}$ $\forall$ n

$\Rightarrow \overline{\mathrm{N}}(S \sim E) = 0. \Rightarrow \mathrm{N}^*(S - E) = 0$]
Part – III: Suppose $\mathrm{N}^*(B) = 0$
Then B is a N^* measurable.
[Since given $\mathrm{N}^*(S) \ge \mathrm{N}^*(S \wedge B) + \mathrm{N}^*(S \wedge B')$
$\ge 0 + \mathrm{N}^*(S \wedge B')$]
By part – II, there is a relative sub lattice $C \in Q_{\sigma\delta} \ni B \subseteq C$ and $\mathrm{N}^*(C \sim B) = 0$.
Now $\mathrm{N}^*(C) = \mathrm{N}^*(C \wedge B) + \mathrm{N}^*(C \wedge B')$
$\Rightarrow \mathrm{N}^*(C) = \mathrm{N}^*(B) + \mathrm{N}^*(C - B)$
$= 0 + 0 = 0$.
Therefore, each relative sub lattice B with $\mathrm{N}^*(B) = 0$ is contained in a relative sub lattice C in $Q_{\sigma\delta}$ with $\mathrm{N}^*(C) = 0$
Hence the theorem

CONCLUSION

After established certain elementary properties of induced outer measure via order preserving valuation on a relative sub lattice, we proved outer measure via order preserving valuation on a relative sub lattice is valid the outer measure generated by measure.

REFERENCES

1. Seshagiri Rao Y.V., Anil Kumar, Characterization of outer measure of Partial lattices in a countable Boolean Lattice, ARPN Journal of Engineering and Applied Sciences, Vol.14, No.4, February 2019, ISSN: 1819-6608.
2. Seshagiri Rao Y.V., Anil Kumar, The Characteristics of Partial Lattices and Irreducibility of Measurable Functions on Partial Lattice, International Journal of Scientific &Technology Research, Vol.8, No.9, September 2019, ISSN: 2277-8616.
3. Szasz Gabor, "Introduction to lattice theory", academic press, NewYork and London,Year 1963.
4. Royden. H.L., Real Analysis, 3rd ed.,Macmillan Publishing, NewYork, Year 1981.
5. Nageswara Rao T., Muneera Abdul, M., Rao, J. V., Srinivas, M. N., & Sateesh Kumar D. (2021). Trends on dominations of fuzzy graphs and anti fuzzy graphs. AIP Conference Proceedings, , 2375 doi:10.1063/5.0066458
6. Lakshmi, K., Mahaboob, B., Sateesh Kumar, D., Balagi Prakash, G., & Nageswara Rao, T. (2021). A new vision on ordinary least squares estimation of parameters of linear model. AIP Conference Proceedings, , 2375 doi:10.1063/5.0066922
7. Lakshmi, K., Mahaboob, B., Sateesh Kumar, D., Balagi Prakash, G., & Nageswara Rao, T. (2021). On mathematical and statistical aspects of linear models AIP Conference Proceedings, , 2375 doi:10.1063/5.0066928